全息原理

Lecturer: 刘 洪

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Emergency of Gravity.

For Strong, weak and elector-magnetic interaction, We can use Gauge theories Based on $SU(3) \times SU(2) \times U(1)$ to describe them, and in principle, We know how to calculate.

But, For gravity. classical gravity = spacetime

quantum gravity = ???

↳ is spacetime the most basically thing?

is spacetime continuous?

1997 Maldacena:

Quantum Gravity = Field Theory.

Dynamic of spacetime

On a fixed space time

means no Gravity

1967. Sakharov There are something can be explained as metric of connection in CMT.

Use Field Theory to describe gravity.

Can we use lower spin particle to compose the massless 2-spin graviton?

Weinberg and Witten 1980 no-go theorem.

① A theory that allows the construction of a

Lorentz Covariant Conserved Current cann't Contain massless particle of spin large than $1/2$ with non-vanishing value of $Q = \int J^0 d^3x$

① A theory that allowed a Lorentz-Covariant Conserved stress Thensor $T^{\mu\nu}$. Can not contain massless particle of spin large than 1.

- QED = Maxwell + Dirac

photon does not carry charge

U(1) J^μ

so photon spin $> \frac{1}{2}$
$= 1$

- In GR there is no Conserved Lorentz-Covariant $T^{\mu\nu}$, $T^{\mu\nu}$ is not conserved in GR, so in (classical) GR, We can have Spin-2 massless particle.

- ② $\Rightarrow$ None of renormalizable QFTs in Minkowski Spacetime can have emergent Gravity.

$\nabla^\mu T_{\mu\nu} = 0 \not\Rightarrow \partial_0 T^{\mu\nu} = 0$ conserved.

To proof these two theorems. We should assume that.

- The particles live in the spacetime is the Original theory.

In Holographic Duality, gravity does not live in the same spacetime.

Proof: one-partical state $|\vec{k},\sigma\rangle$, $\sigma=\pm j$ ← helicity.

$\hat{R}(\theta,\hat{k})$. SO(3) operator. arround axis $\hat{k}=\frac{\vec{k}}{|\vec{k}|}$

$\hat{R}(\theta,\hat{k})|\vec{k},\sigma\rangle = e^{i\sigma\theta}|k,\sigma\rangle$ (~~Def~~ of Helicity)

$\hat{P}^{\mu} = \int d^3x\, T^{0\mu} \Rightarrow \hat{P}^{\mu}|k,\sigma\rangle = k^{\mu}|k,\sigma\rangle$

$\hat{Q}|k,\sigma\rangle = q|k,\sigma\rangle$

(i) Lorentz Symmetry.

$$\langle k,\sigma|\hat{J}^{\mu}|k',\sigma\rangle \xrightarrow{k\to k'} \frac{q k^{\mu}}{k^0}\cdot\frac{1}{(2\pi)^3} \qquad ①$$

$$\langle k,\sigma|\hat{T}^{\mu\nu}|k',\sigma\rangle \xrightarrow{k\to k'} \frac{k^{\mu}k^{\nu}}{k^0}\cdot\frac{1}{(2\pi)^3} \qquad ②$$

where $\langle k,\sigma|k',\sigma'\rangle = \delta_{\sigma\sigma'}(2\pi)^3\delta^{(3)}(k,k')$

To Derive ①. you just need to prove $\langle k,\sigma|J^0|k',\sigma\rangle = \frac{q}{(2\pi)^3}$

Similarly. you can prove Eq. ②

(ii) Massless. $\Rightarrow k^2=0$, $k'^2=0$

$\Rightarrow k\cdot k'<0$. $\Rightarrow k+k'$ is time-like

So you can choose a fram that $\vec{k}+\vec{k}'=0$

$k=(E,0,0,E)$, $k'=(E,0,0,-E)$

(iii) Under a Rotation around $\hat{z}$.

$\hat{R}_z(\theta)|k,\sigma=+j\rangle = e^{i\cdot j\cdot\theta}|k,j\rangle$

$\hat{R}_z(\theta)|k',\sigma=+j\rangle = e^{-i\cdot j\cdot\theta}|k',j\rangle$

Because $\vec{k},\vec{k}'$ have opposite derictions

Consider $\langle k', j | \hat{R}^{-1}(\theta) J^\mu \hat{R}(\theta) | k, j \rangle$

$$= e^{2ij\theta} \langle k', j | J^\mu | k, j \rangle$$

$$= \Lambda^\mu{}_\nu(\theta) \langle k', j | J^\nu | k, j \rangle$$

But $\Lambda^\mu{}_\nu(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \Lambda^\mu{}_\nu(\theta)$ has eigenvalue $e^{2ij\theta}$

But $\Lambda^\mu{}_\nu(\theta)$ can only has eigenvalue $e^{\pm i\theta}, 1$

so $j = 0, \frac{1}{2}$, so $q \neq 0 \Rightarrow j = 0, \frac{1}{2}$.

Similarly. $e^{2ij\theta} \langle k', j | T^{\mu\nu} | k, j \rangle$

$$= \Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho \langle k', j | T^{\lambda\rho} | k, j \rangle$$

$\Rightarrow j \leq 1$

Classical Black Hole

Important scales.

- $m_p = \sqrt{\frac{\hbar c}{G_N}} \simeq 1.2\times10^{19}\,\mathrm{GeV}/c^2$
- $l_p = \sqrt{\frac{\hbar G_N}{c^3}} \simeq 1.6\times10^{-33}\,\mathrm{cm}$

Strength of Gravity.

- For Electro-Magnetic, Compton Wavelength: $r_c = \frac{\hbar}{mc}$
  this tell us to what extend we can call a single particle, the effective strength of EM:

$\Rightarrow \lambda_e = \frac{V(r_c)}{mc^2} = \frac{e^2}{\hbar c} = \alpha = \frac{1}{137}$, this is the coupling const of QED

- For Gravity $V = -\frac{G_N m^2}{r}$

$$\lambda_g \equiv \frac{V(r_c)}{mc^2} = \frac{G_N m^2}{\hbar/mc} \cdot \frac{1}{mc^2} = \frac{G_N m^2}{\hbar c} = \frac{m^2}{m_p^2} = \frac{l_p^2}{r_c^2}$$

$m \ll m_p \Rightarrow \lambda_g \ll 1$ for $e^- \Rightarrow \frac{\lambda_g}{\lambda_e} \sim 10^{-43}$

Gravity really weak!

Schwarzchild Radius

$$\frac{G_N m m'}{r_s} \sim m'c^2 \Rightarrow r_s \sim \frac{G_N m}{c^2}$$

At r_s. the classical Gravity should be replaced by Einstein Gravity

From the analysis of GR $r_s = \frac{2G_N M}{c^2}$ is the size of a black hole.

Remarkable thing: Black Hole can Have Quantum Effect Manifest at Macroscopic level at length scales $O(r_s)$, Although $m \ll m_p$

Without Gravity: $|\vec{p}| \to \infty$, length sale $\to 0$

With Gravity: when $E_{cm} \gg m_p$, r_s dominate since $r_s \propto E$. So $|\vec{p}| \uparrow$. $r_s \uparrow$ length scale $\uparrow$. So Planck length is the minimal length scale one can probe Because the E too large. you create a black Hole

Various Regimes of gravity.

- classical gravity: $\hbar \to 0$, G_N finite.
- QFT in a fixed spacetime: $\hbar$ finite $G_N \to 0$ (perturbative Quantune Gravity)
- Quantum Gravity: G_N. $\hbar$ finite.
- semi-classical Regime of Quantum Gravity: keep $\hbar$ finite, expand in G_N (arrond $G_N = 0$)

AdS/CFT

Schwarz Metric. ($\Lambda=0$)

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$f = 1 - \frac{2G_N M}{r} = 1 - \frac{r_s}{r}$$

event horizon: $r = r_s$, where $g_{tt} = 0$, $g_{rr} = \infty$

When $r < r_s$, f switches sign

Some simple facts:

① time-reversal invariant.

- so the Schwarz Metric only can describe the stablized Black Hole After collapse

This is the Mathematical idealization of the BH.

② There is no spacetime singularity at $r = r_s$.

this is the coordinate singularity

③ $r = r_s$ is a null Hypersurface : have null geodesic.

④ The horizon is a surface of infinite redshift.

Consider an observer O_h at $r = r_h \simeq r_s$.

- . . . . O_∞ at $r = \infty$

at $r = \infty$ $ds^2 = -dt^2 + dr^2 + \cdots$ (AFS.)

t: propertime of O_∞

at $r = r_h$. $ds^2 = -f(r_h)dt^2 + \cdots$

$d\tau = \sqrt{f(r_h)}\,dt = \sqrt{1 - \frac{r_s}{r_h}}\,dt$.

as $r_h \to r_s \Rightarrow d\tau_h/dt \to 0$. Compare to the time at $r = \infty$ the time at $r \approx r_s$ becomes infinitely slow

$E_\infty = E_h f^{\frac{1}{2}}(r_h)$. i.e. for any finite E_h, $E_\infty \to 0$ as $r_h \to r_s$. so infinitely redshift. Any event at O_h. has no energy by the view of O_∞.

④ It takes a "free-fall" traveler a finite proper time. to reach the horizon, but. infinite Schwarzchild time "Frozen at the horizon"

⑤ Once inside the horizon, a traveler can not send Signals to outside. nor can escape.

⑥ Area of a spatial section:

$$\boxed{A_h = 4\pi r_s^2 = 16\pi G_N^2 M^2}$$

Surface gravity: (Defined by the acceleration of a stationary observer at the horizon as measured at infinity.

$$\boxed{\kappa = \frac{1}{2} f'(r_s) = \frac{1}{2r_s} = \frac{1}{4G_N M}}$$

Causal Structure of Black Hole

Consider the region near the horizon, $r \gtrsim r_s$

$$f(r) = f'(r_s)(r - r_s) + \cdots$$

Consider the proper distance ρ from the horizon

$$d\rho = \frac{dr}{\sqrt{f}} \overset{\text{near } r_s}{=\!=} \frac{dr}{\sqrt{f'(r_s)(r-r_s)}} + \cdots$$

$$\Rightarrow \rho = \frac{2}{\sqrt{f'(r_s)}} \sqrt{r - r_s} + \cdots \Rightarrow f(r) = \underbrace{\left[\tfrac{1}{2} f'(r_s)\right]^2}_{\kappa^2} \rho^2$$

↑ measured by observer at horizon

Integral constant is $\rho(r_s) = 0$

$$\Rightarrow ds^2 = -\kappa^2 \rho^2 dt^2 + d\rho^2 + r_s^2 d\Omega_2^2 + \mathcal{O}(r - r_s)$$

so we transform to the reference from of the observer at the horizon (stationary)

Define: $\eta = \kappa t \equiv \frac{t}{2r_s}$

$$\Rightarrow ds^2 \approx \underbrace{-\rho^2 d\eta^2 + d\rho^2}_{(1+1)\text{d Minkowski spacetime (In Rindler Form)}} + \underbrace{r_s^2 d\Omega_2^2}_{S^2 \text{ with Radius } r_s}$$

So At Horizon the spacetime of Schwarz spacetime is $(1+1)$d Rindler $\times S^2$

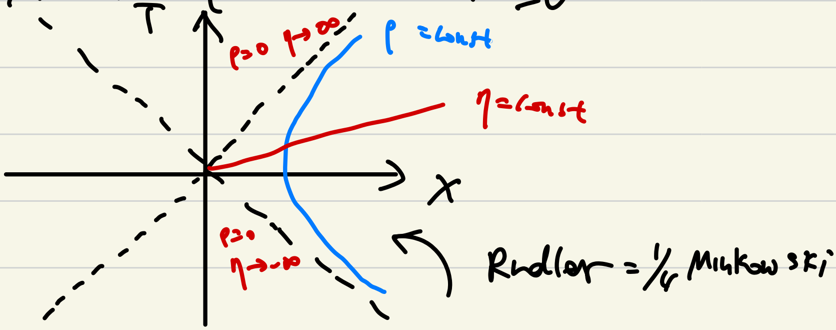
Minkowski $\quad ds^2 = dT^2 + dX^2$

Introduce $X = \rho \cosh\eta, \quad T = \rho \sinh\eta$

$\Rightarrow \quad ds^2 = -\rho^2 d\eta^2 + d\rho^2$ So Rndler $\tilde{=}$ Minkowski

But Rndler spacetime has Event Horizon, It only Cover $1/4$ of Minkowski spacetime

Because. $X^2 - T^2 = \rho^2$ and $X \geq 0$



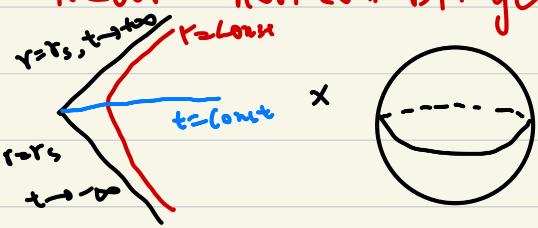
Accelarating observer can only see a part of Our Universe

$\Rightarrow \rho = 0 \Rightarrow X = \pm T.$

On the horizon (not near, exactly at)

So BH Horizon Corresponds to $X = \pm T$

$\Rightarrow$ near - horizon BH geometry = Rndler $\times S^2$



1) An observer at $r = \text{const} \Leftrightarrow \rho = \text{const}$ in Rindler. So although he stay in Schwarz spacetime, he actually accelerating in Minkowski spacetime patch.

$\hookrightarrow a = \frac{1}{\rho} = \frac{1}{2}\sqrt{f'(r_s)} \times \frac{1}{\sqrt{r-r_s}}$ $\quad a^\mu = u^\nu \nabla_\nu u^\mu,\ u^\mu = \frac{dx^\mu}{d\tau}$

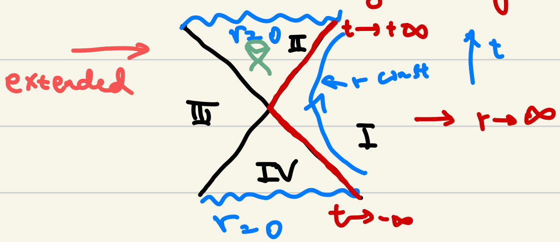
↑ viewed by himself, k is the a viewed by infinity

$\Rightarrow a_\infty = a(r) f^{\frac{1}{2}}(r) = k$

2) free-fall observer near the BH horizon also free-fall in M_2, so it is the inertial from in M_2

3) Rindler spacetime is singular at $\rho = 0$, like $r = r_s$ it's the coordinate singular, But $\exists$ coordinate we can Extend Rindler space time to Minkowski spacetime and remove this singularity.

Similarly, by changing to suitable coordinate (Kruskal) one can extend the schwarz Geometry to full Regions. Remove the Singularity at $r = r_s$



time ↔ space

(a) fall into BH, I ⟶ II

(b) IV $\underset{\text{Reverse}}{\overset{\text{time}}{\longleftrightarrow}}$ II, for real BH III and IV do not exist

(c) $r = 0$ is the BH Singularity

Black Hole Temperature

In QFT, to describe a system at finite T, we need go to Euclidean $t \to -i\tau$ $\tau \sim \tau + i\beta$, $\beta = \frac{1}{T}$

Conversely. if $t \to -i\tau$, τ is periodicly then, this system is finite Temperature.

For BH, $t \Rightarrow -i\tau \Rightarrow ds_E^2 = f d\tau^2 + \frac{1}{f} dr^2 + r^2 d\Omega_2^2$

near the horizon. $ds_E^2 = \rho^2 \kappa^2 d\tau^2 + d\rho^2 + r_s^2 d\Omega_2^2$

Introduce $\theta = \kappa\tau \Rightarrow ds_E^2 = \rho^2 d\theta^2 + d\rho^2 + r_s^2 d\Omega_2^2$

This metric has a conical singularity at $\rho = 0$ unless $\theta \sim \theta + 2\pi$, since the horizon non-singular in the Lorentz picture, it should not be singular in Euclidean. $\Rightarrow \theta \sim \theta + 2\pi \Rightarrow \tau \sim \tau + \underbrace{\frac{2\pi}{\kappa}}_{\beta}$

$\Rightarrow \beta = \frac{2\pi}{\kappa} \Rightarrow T = \frac{\kappa}{2\pi}$ Recall $t = -i\tau$, t is the proper time of $r = +\infty$ observer so $r = \infty$ observer will feel temperature at

$$T_\infty = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{4\pi r_s} = \frac{\hbar}{8\pi GM}$$

For an observer at some r (stationary! at r! not fall into)

$$dt_{loc} = f^{\frac{1}{2}}(r)\, dt \Rightarrow T_{loc}(t) = T_\infty f^{-\frac{1}{2}}(r)$$

at r_s $T \to \infty$

Similar. For Rinderler spacetime

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \xrightarrow{\eta = i\theta} \rho^2 d\theta^2 + d\rho^2$$

$\Rightarrow \theta \sim \theta + 2\pi$, For Rindler observer at constant ρ

propertime. $dt_{loc} = \rho d\eta \Rightarrow d\tau_{loc} = \rho d\theta$

$$\tau_{loc} \sim \tau_{loc} + \underline{2\pi\rho}$$

$\leftarrow \hbar\beta$

$$T_{loc}(\rho) = \frac{\hbar}{2\pi\rho} = \frac{\hbar}{2\pi} \cdot a_{loc} \quad (a_{loc} = \frac{1}{\rho})$$

Unruh Effect!

Physics interpratation of Temprature

Consider QFT in BH spacetime. The Vacuum state obtained via the analytic continuation procedure from Euclidean signature is a thermal equilibrium stated temperature (depends on the proper time w.r.t. the observer)

①: The choice of Vaccum for a QFT in a curved spacetime is not unique. It's observe dependent. So. when we use the Wick Rotation to

Calculate the Temperature. We make a special choice of the Vacuum:

For BH : "Hartle - Hawking" Vacuum

For Rindler : Minkowski Vacuum reduced to the Rindler Patch

② If for a BH, we take τ to be uncompact.

Wick Rotation back to Lorenz.

$\Rightarrow$ Schwarzchild (Boulware) Vacuum

This is the Vacuum that one would get by doing Canonical quantization in a BH in terms of t

For Rindler : θ uncompact $\Rightarrow$ Rindler Vacuum.

Can also be obtained by canonical quantization

③ In Schwarzchild Vacuum, the corresponding Euclidean manifold is singular at $r = r_s$.

$\Rightarrow$ physical observables are also singular at r_s

so H-H Vacuum is more physical than S Vacuum

Physical Origin of the Temprature

Two Equivalent Ways to describe statical mechanics

① $\langle O \rangle = \frac{1}{Z} \mathrm{Tr}(e^{-\beta H} O)$

② Umezawa 1960's

consider double copy of the same system

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |n_1\rangle \otimes |n_2\rangle \in \mathcal{H}$$

For any observable O_1 in $\mathcal{H}_1$

$$\langle \Psi | O_1 | \Psi \rangle = \frac{1}{Z} \sum_n e^{-\beta E_n} \langle n | O_1 | n \rangle = \langle O_1 \rangle$$

$$\mathrm{Tr}_2(|\Psi\rangle\langle\Psi|) = \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle_1 {}_1\langle n|$$

$$= \rho_1$$

From pure state to mixture state

Temperature arises due to our ignorance of System 2. $\mathcal{H}_2$

For Harmonic Oscillators. $\underbrace{|\Psi\rangle = \frac{1}{\sqrt{Z}} \exp\left[e^{-\frac{\omega\beta}{2}} a_1^\dagger a_2^\dagger\right] |0\rangle_1 |0\rangle_2}_{\text{Squeeze state}}$

Define $b_1|\Psi\rangle = b_2|\Psi\rangle = 0$

$$\Rightarrow \quad b_1 = \cosh\theta\, a_1 - \sinh\theta\, a_2^\dagger$$

$$b_2 = \cosh\theta\, a_2 - \sinh\theta\, a_1^\dagger$$

Bogoliubov Transformation

$$\cosh\theta = \frac{1}{\sqrt{1-e^{-\beta\omega}}}, \quad \sinh\theta = e^{-\frac{1}{2}\beta\omega}\cosh\theta .$$

$|0\rangle_1 \otimes |0\rangle_2$ is Vacuum for a_1, a_2

$|\Psi\rangle$ is Vacuum for b_1, b_2

Schrödinger representation of QFT.

configuration space : $\phi(\vec{x}, t=0) := \phi(\vec{x})$ on the spatial slice

Hilbert space : $\mathcal{H} = \{\Psi[\phi(\vec{x})]\}$, Functional

$$\langle \phi_2(\vec{x}), t_2 | \phi_1(\vec{x}), t_1\rangle = \int_{\phi(\vec{x}, t_1) = \phi_1(\vec{x})}^{\phi(\vec{x}, t_2) = \phi_2(\vec{x})} \mathcal{D}\phi \, e^{iS[\phi]}$$

Vacuum wave functional

$$\langle \underset{t_2 = 0}{\phi(\vec{x})} | \underset{t_1 = -\infty}{0}\rangle = \Psi_0[\phi(\vec{x})] = \int_{t_E<0}^{\phi(\vec{x}, t_E = 0) = \phi(\vec{x})} \mathcal{D}\phi(\vec{x}, t_E)\, e^{-S_E[\phi]}$$

$t_E := \mathrm{Im}\, t$ Integrate for all $t_E < 0$ why?

$|0\rangle$ means no boundary at t_1. $(1+\varepsilon)H$ trick (Srednicki)

$H \to (1-i\epsilon)H$ then:

$$|\phi'(\vec{x}),t\rangle = e^{iHt}|\phi'(\vec{x})\rangle \xrightarrow{t\to-\infty} \langle 0|\phi'(\vec{x})\rangle |0\rangle$$

← Heisenberg $t=0$ configuration space

$$\Rightarrow |0\rangle = \lim_{t\to-\infty} \int d\phi'(\vec{x})\, |\phi'(\vec{x}),t\rangle$$

$$\Rightarrow \langle\phi(\vec{x})|0\rangle = \int d\phi' \langle\phi(\vec{x}),t=0|\phi'(\vec{x}),-\infty\rangle$$

$$= \int d\phi' \int_{\phi(\vec{x},t=-\infty)=\phi'}^{\phi(\vec{x},t=0)=\phi(\vec{x})} D\phi \; e^{iS(\phi)}$$

$$iS(\phi) = i\int_{-\infty}^{0} dt\,\mathcal{L}$$

$$= \int^{\phi(\vec{x},t=0)=\phi(\vec{x})} D\phi \; e^{iS(\phi)} \leftarrow H \mapsto (1-i\epsilon)H$$

So $|0\rangle$ means no boundary

Quantum mechanics, analog

$$\langle x|0\rangle = \psi_0(x) = \int^{x(t_E=0)=x} Dx(t_E)\, e^{-S_E[x]}$$

$$S_E = \int_{-\infty}^{t_E=0} dt_E\, L$$

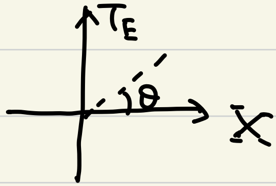
Come back to a QFT in Rindler spacetime.

$$ds^2 = -dT^2 + dX^2 \underset{1/4}{=\!=} -\rho^2 d\eta^2 + \rho^2$$

Mink can see $\}$ and $\{$, but Rind can only see the $\{$

$|0\rangle_M$

$T \mapsto iT_E$, $\eta \mapsto i\theta$. With $\theta \sim \theta + 2\pi$.



When you going to Euclidean. Signature

Mink = Rindler = $\mathbb{R}^2$. !

$\Rightarrow$ Euclidean observables are identicle. ①

For Mink: back to Lorentzian.

② Euclidean functions $\Rightarrow$ correlation function in Mink

①+② For Rindler $\Rightarrow$ correlation function in Mink Vacuum

But for observables restricted to Rindler patch.

Back to Lorentzian:

$\mathcal{H}_{Rindler} = \{\Psi(\phi_R)\}$ ϕ_R is $\phi(X>0, T=0)$

Right Restrict to Rindler Patch.

Hamiltonian : H_R. $\Rightarrow$ spectrum is $|n\rangle_R$.

$|0\rangle_R$ Called Rindler Vacuum.

Warning. H_R is Lorentzian. quantized by η. not by periodic $\theta \sim \theta + 2\pi$.

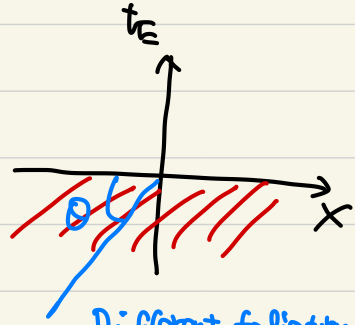
$\mathcal{H}_{mink.} = \{\Psi[\phi(X)]\}$. H_M, $|n\rangle_M$, $|0\rangle_M$.

$\Psi_0[\phi(X)] = \langle\phi(X)|0\rangle_M := \Psi_0[\phi_L(x), \phi_R(x)]$

$\phi_L(x)$, $\phi_R(x) \Rightarrow \mathcal{H}_{Mink} = \mathcal{H}_L^{Rind} \otimes \mathcal{H}_R^{Rind}$

①+② tell us $\mathcal{H}_R$ and $\mathcal{H}_M$ both have the same Ψ. but for Rindler should be Restricted to $\phi_R \subset \phi$

$$\Psi_0[\phi(x)] = \int_{t_E<0}^{\phi(x,t_E=0)=\phi(x)} D\phi(t_E, x)\, e^{-S_E}$$



Different foliation

θ as time

$$= \int_{\phi(\theta=-\pi,\rho)=\phi_L(x)}^{\phi(\theta=0,\rho)=\phi_R(x)} D\phi(\theta,\rho)\, e^{-S_E}$$

$$= \langle \phi_R(x), \theta=-\pi \,|\, \phi_L(x), \theta=0 \rangle$$

$$= \langle \phi_R |\, e^{-i(-i\pi)H_R} \,| \phi_L \rangle$$

imaginary time evolution

$$= \sum_n e^{-\pi E_n} \chi_n[\phi_R]\, \chi_n^*[\phi_L]$$

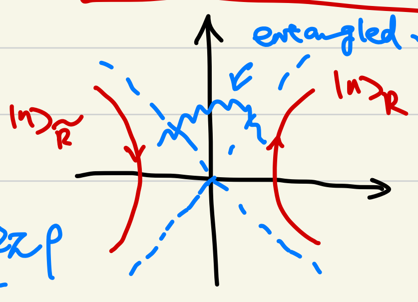
$\chi_n[\phi] := \langle \phi | n \rangle_R \in \mathcal{H}_R$.

$\chi_n^*[\phi] \in \tilde{\mathcal{H}}_R$, $\mathcal{H}_R$ with opposite direction.

full with particles

$$\boxed{|0\rangle_M = \sum_n e^{-\pi E_n} |n\rangle_{Rind} \otimes |n\rangle_{\widetilde{Rind}}}$$

$d\tau_{loc} = \rho\, d\eta$



$\Rightarrow \beta_{loc} = 2\pi\rho$

trace out left Rindler

$$\mathrm{Tr}_L\left(|0\rangle_M \langle 0|\right) = \rho_{Rind} = \frac{1}{Z_{Rind}} e^{-2\pi H_R}$$

$\Rightarrow \beta = 2\pi$. (associate with η)

① $|0\rangle_M$ is invariant under $H_R^{(R)} - H_R^{(L)}$

come from left patch running in opposite direction

Invariant means. $e^{-i\eta(H_R^{(R)} - H_R^{(L)})} |0\rangle_M = |0\rangle_M$

as generator. $\eta \mapsto -\eta$ in opposite direction

Geometrically, η translation is a boost in (X, T) Minkowski Coordinate. so H_R generator a boost $\Rightarrow$ $|0\rangle_M$ is invariant under boosts

tips: Recall $X = \rho \cosh\eta$ $T = \rho \sinh\eta$ so $\eta \mapsto \eta + d\eta$ means boost in X, T. η is the Rapidity.

② If we expand ϕ_R in terms of a complete set of Modes in Right patch:

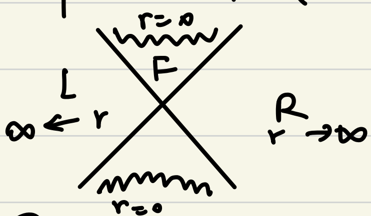
$$\phi_R = \sum_j \left(a_j^R u_j + a_j^{R\dagger} u_j^*\right), \quad a_j^R |0\rangle_R = 0$$

similarly do this on ϕ_L.

$$|0\rangle_M = \frac{1}{\sqrt{Z_{Rind}}} \prod_j \exp\left(e^{-\pi\omega_j} a_{jR}^\dagger a_{jL}^\dagger\right) |0\rangle_R \otimes |0\rangle_L$$

Usually Minkowski creation and annihilation are related to a_{jR} and a_{jL} by Bogolinbov transformation just as the harmonic oscillator example. (See Curved Spacetime QFT textbook)

③ All the discussion generalizes in complete parallel to Schwarzchild spacetime



$|0\rangle_{HH}$ is the Entangled state between L and R.

④ BH formed by gravitational collapse only have R and F regions. You don't have Left Region!. Left Patch is the Kaluza Extension Mathematically. So this pach doesn't work.

So we need a different explaination for T_{BH} of "Real" Balack Hole. local observer can not feel difference between L+R and R alone so we can use this Method to explain temperature observed by local observer.

Further Example: Kerr-Newman BH.

Generally, spherical BH metrics:

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + \dots$$

Horizon: $r = r_0$ $\quad f(r_0) = h(r_0) = 0$

near Horizon

introduce $d\rho = \frac{dr}{\sqrt{h(r)}} \Rightarrow \rho(r) = \frac{2}{\sqrt{h'(r_0)}}\sqrt{r-r_0} + \cdots$

$f(r) = f'(r_0)(r-r_0) + \cdots \sim \frac{1}{4}\rho^2 h'(r_0) f'(r_0)$

the near Horizon geometry is Rindler $\subset$ Mink$_D$. $t \mapsto -i\tau$

$$\Rightarrow ds^2 = \rho^2\left(\frac{\sqrt{h'(r_0)f'(r_0)}}{2}d\tau\right)^2 + d\rho^2 + \cdots$$

It will be revisited later

like we did before, compare this metric to $d\rho^2 + \rho^2 d\theta^2$

$d\theta = \frac{\sqrt{h'(r_0)f'(r_0)}}{2}d\tau$, $\theta \sim \theta + 2\pi \Rightarrow \tau \sim \tau + \frac{4\pi}{\sqrt{h'(r_0)f'(r_0)}}$

Periodic in spacetime means finit temperature QFT.

$$\boxed{T_H = \frac{\sqrt{h'(r_0)f'(r_0)}}{4\pi}}$$

Apply this formulae to Kerr metric:

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma}dt^2 + \frac{\Sigma}{\rho^2}\sin^2\theta(d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where : $\rho^2 = r^2 + a^2\cos\theta^2$, $\Delta = r^2 + a^2 + Q^2 - 2Mr$, $a \equiv \frac{J}{M}$

$\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$, $\omega = \frac{a}{\Sigma}(r^2+a^2-\Delta)$

$G_N := 1$.

two Horizon: $r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}$, We now only consider r_+

temperature : $T_H = \frac{2(r_+ - M)}{A}$

(Outer) Horizon Area : $A = 4\pi(r_+^2 + a^2)$ attention, the induced metric is $\gamma = \frac{\Sigma}{\rho^2}\sin^2 d\phi^2 + \rho^2 d\theta^2$, $A = \int d\theta d\phi \sqrt{\gamma}$. You may be naive think $A = 4\pi r^2$. This is NOT the spherical coordinate.

Angular velocity: $\Omega := \left.\frac{d\phi}{dt}\right|_{ds=dr=d\theta=0} = \omega(r_+) = \frac{4\pi a}{A}$

$r_+ > 0 \Rightarrow M^2 \geq a^2 + Q^2$ if the inequality is saturated two horizons are merging to each other.

$$r_+ = r_- = M$$

We call the BH as extremal BH

$T = 0$, $S = \frac{A}{4} = \pi(M^2 + a^2) \neq 0$ (entropy will be introduced in next section). Now consider its near horizon geometry. Consider a simpler case. $a=0$, $r_0 = M = Q$

$$ds^2 = -\frac{(r-Q)^2}{r^2}dt^2 + \frac{r^2}{(r-Q)^2}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\simeq \underbrace{-\frac{\xi^2}{Q^2}d\tau^2 + \frac{Q^2}{\xi^2}d\xi^2}_{AdS_2} \times \underbrace{+ Q^2(d\theta^2 + \sin^2\theta\, d\phi^2)}_{S^2} + \cdots$$

where $r - Q := \lambda\xi$, $\tau := \lambda t$, $\lambda \to 0$

So for extremal BH, the near horizon geometry is $AdS_2 \times S^2$, not $Rind_2 \times S^2 \subset Mink_2 \times S^2$ which we discussed for Schwarzschild BH.

Black Hole Thermodynamics

BH Not only has temperature, it also obey thermodynamics,

$$T_{BH} = \frac{\hbar \kappa}{2\pi} = \frac{\hbar}{8\pi G_N M}$$

$$\frac{dS}{dE} = \frac{1}{T(E)} = \frac{8\pi G_N M}{\hbar} \quad (E = M)$$

S is just a defination as now.

$$\Rightarrow S(E) = \frac{4\pi G_N E^2}{\hbar} = \frac{4\pi r_s^2}{4\hbar G_N} = \frac{A_{BH}}{4\hbar G_N}$$

$$M \uparrow \Rightarrow T_{BH} \downarrow \Rightarrow C_{BH} = \frac{\partial E}{\partial T} < 0$$

Interlude: General Black Hole

No hair theorem: a stationary, asymptotically Flat Black Hole is fully chacterized by Its mass, angular momentum and conserved gauge charges (e.g. electric charge in strry theory R-R charge)

$$\text{Star} \xrightarrow{\text{Collapse}} \text{BH}$$ All features of star will be lost

Four laws of BH mechanics

0-th law: surface gravity K is constant over the horizon

$$\text{1st law:}\quad dM = \underbrace{\frac{\kappa}{8\pi G_N} dA}_{Tds} + \Omega\, dJ + \Phi\, dQ$$

angular frequency   electrical potential (gauge)

2nd law: Horizon area never decreases

$\Delta S \geq 0$

3rd law: Surface gravity κ cannot be reduced to zero in a finite number of procedures

$\kappa > 0$

These laws and the formulae of T_{BH} and S_{BH} are not only for Schwarchild Hole. but also every BHs. and they can be deduced by classical GR. Historically, before Hawking's discovery of Black Hole radiation Bekenstein think. the thermodynamics is more fundmental, so He gass BH should be a thermodynamics object and proved if $S_{BH} \propto A$, than $\Delta S_{tot} \geq 0$. $S_{tot} = S_{BH} + S_{matter}$ so when you throw an object into BH the total entropy will not decrease because BH is a thermodynamic object and has entropy. Hawking Radiation prove that BH is Realy a thermodynamic object.

Example: Kerr-Newman BH does obey these 3 laws.

1) Few pages ago, we found expressions for A, Ω, S and T

now, use da, dM, dQ to express their differential:

$$dS = \frac{4\pi(r_+ - M)}{A}\left(r_+ dM + a\,da + \frac{r_+}{r_+ - M}(M\,dM - a\,da - Q\,dQ)\right)$$

$$dJ = a\,dM + M\,da$$

It's straight forward to obtain: 1st law

$$\boxed{T dS + \Phi\, dQ + \Omega\, dJ = \frac{4\pi}{A}(r_+^2 + a^2)\,dM = dM}$$

2) the proof of 2nd and 3rd law need more technical steps, I will not show it here, but we can image a Simpler case to explain the 3rd law. Consider KN-BH with $J=0$, $M>Q$, we can put in particle with charge-mass ratio $\frac{q}{m} > 1$ to make Q to be more closed to M, i.e., $T \to 0$. In order for a charged particle to be able to fall into the BH, Coulomb Force should less than Gravitational Force, so $qQ \le Mm$:

$$\boxed{1 < \frac{q}{m} \le \frac{M}{Q}}$$

Assume we were close to extremality to begin with and expand $\frac{Q}{M}$ near 1

next step $\Rightarrow$ $$\frac{Q'}{M'} = \frac{Q+q}{M+m} = \frac{Q}{M}\cdot\frac{1+\frac{M^2}{Q^2}\frac{m}{M}}{1+\frac{m}{M}} = 1 - \left(1-\frac{Q}{M}\right)\frac{1-\frac{m}{M}}{1+\frac{m}{M}} + \cdots$$

For any $\frac{m}{M} < 1$, $\frac{Q}{M} < 1$, we obtain $\frac{Q'}{M'}$ also less than 1.

So we can not cool the BH by finite steps (3rd law)

1) Does BH entropy have a statistical interpretation?

so BH should have micro states

$$N \sim e^{S} = e^{\frac{A_{BH}}{4\hbar G_N}}$$

2) Information loss paradox: During Hawking Radiation Information lost. because Radiation is information - free

Holographic principle.

Consider isolated system in AFS (Asymptotic flat spacetime) with mass E and entropy S_0.

A: area of the sphere that enclosed the system

M_A: mass of the BH with horizon area A.

$\Rightarrow$ $E \le M_A$, because the system don't collapse to a black hole yet

Add $M_A - E$ amout of Energy to the system (Keeping A fixed). Then we creat a Black Hole

so $S_{BH} \geq S_0 + S'$ (2nd law)

Entropy of added energy.

$$\Rightarrow \quad S_0 \leq S_{BH} = \frac{A}{4\hbar G_N}$$

$\Rightarrow$ Maximal Entropy inside a region bounded by area A is given by:

$$\boxed{S \leq \frac{A}{4\hbar G_N}}$$

Von. Neumann Entropy:

$$S = -\mathrm{Tr}\,\rho \ln \rho.$$

For finit N-dimensional Hilbert space.

$$S_{max} = \log N. \qquad (\rho = \frac{1}{N} \mathbb{1})$$

$$\Rightarrow \quad \log N \leq \frac{A}{4\hbar G_N} \quad \Rightarrow \quad N \leq \exp\left(\frac{A}{4\hbar G_N}\right)$$

N is the effective dimension of the system.

even for harmonical oscillator, $\dim \mathcal{H} = \infty$
but for a system with finit number of degrees of fredom, Below some energy scale. the $\mathcal{H}_{eff}$ is always finite dimension.

In typical. physical systems:

$$\#\text{ of d.o.f.} \sim \log N_{eff} \leq \frac{A}{4\hbar G_N}$$

But # of d.o.f is propertional to the Volume. not Area! so this bound is certainly violated in non-gravitational systems.
Because only with gravity. things will collapse. to a black hole. So we can't creat a huge mass in a fixed Area.
Quantum Gravity leads to a huge reduction of # of d.o.f.
so in quantum gravity. a region of boundary area A can be fully described by #d.o.f. no more than:

$$\#. d. o. f \leq \frac{A}{4\hbar G_N} = \frac{A}{4 l_p^2}.$$

This is the Holographic principle. it tells you that in Quantum gravity. Informasion is encoded on the surface not in the volume.

Large N matrix theories.

Consider SU(N) YM theory

$$\mathcal{L} = \frac{1}{4 g_{YM}^2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$$

$A_\mu = A_\mu^a T^a$. $T^a \in su(N)$. a Matrix.

take $N \to \infty$ limit, do $\frac{1}{N}$ expansion.

$SU(\infty) \longleftrightarrow$ String Theory

Srednicki

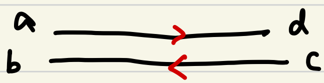
Consider more simpler scalar matrix Field theory

$$\mathcal{L} = -\frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{4}\Phi^4\right]$$

$\Phi(x) = \Phi^a{}_b(x)$, $(\Phi^a{}_b)^* := \Phi^b{}_a \in u(N)$

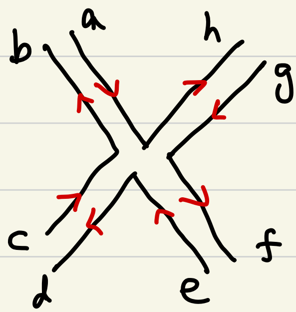
$U(N)$ is the global symmetry. We don't introduce gauge field to make it local.

Feynman Rules: Doubline notation.



$$= \frac{g^2\delta^a{}_d\,\delta^c{}_b}{p^2} \sim \frac{g^2}{p^2}$$

→ up to low indices

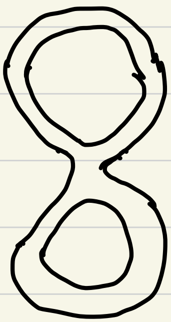


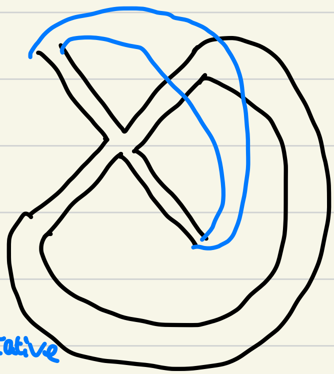
$$= \frac{1}{g^2}\delta_b{}^c\,\delta_d{}^e\,\delta_f{}^g\,\delta_h{}^a \sim \frac{1}{g^2}$$

Vacuum energy.

planar

non-planar.





every loop

$\Rightarrow$ Trace Matrix δ^a_b

$\Rightarrow N$.

$\Downarrow$ $N^3 g^2$

because matrices non-commutative

$\Downarrow$ $\neq$

$\Downarrow$ Ng^2

We omitt the momenta integrals

You can always draw non-planar diagram on a genus g surface without crossing. Assume we have done.

<u>General formulae:</u>

Power of N = # of faces

$\Rightarrow$ For Vacuum diagram:

$$\text{Amp} \sim (g^2)^E \cdot (g^{-2})^V \cdot N^F$$

$$g^{2(E-V)} \cdot \frac{\lambda^F}{g^{2F}}$$

$$\sim g^{2(E-V-F)} \cdot g^{2F} \cdot N^F$$

$$L = E - V + 1$$

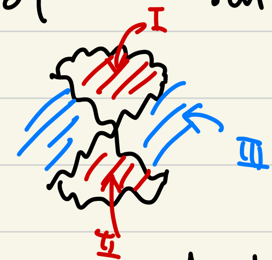
the # of loops

$$\sim g^{-2\chi} \cdot (g^2 N)^F \Rightarrow \boxed{\lambda^{L-1} N^{\chi}}$$

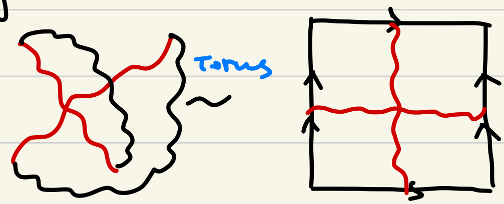
Large N. and <u>keeping $g^2 N = \text{const} := \lambda$</u>

sphere $\chi = 2$ Torus $\chi = 0$ ...... 't Hooft Limit

Each diagram can be considered as the partition of the surface into polygons



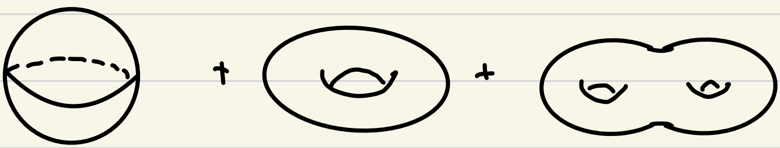
partition on sphere



As before, we know. this to diagram is different For non-Abelian Gauge Theory

That's why $\chi = F + V - E = 2 - 2g$.

$SU(\infty) =$



N^2 $\qquad N^0$ $\qquad N^{-2}$

Recall $Z = e^{iW}$

$$\log Z_0 = N^2\left(C_0 + C_1\lambda^1 + C_2\lambda^2 + \cdots\right) \quad \text{planar}$$

$$+ N^0\left(\cdots\cdots\cdots\cdots\right) + \cdots \quad \text{non-planar}$$

counting the # of Loops

expand by Topology!

Actually. this series is convergent. not just an asymptotic expansion.

W is the summand of all connected diagrams. $Z = \int \mathcal{D}\phi\, e^{iS}$. is the Vacuum Energy. only Connected diagrams have no-trivial contribution for LSZ formulae.

Although we use ϕ^4 theory to be the example. But actually for any Lagrangian of matrix valued fields:

$$\mathcal{L} = \frac{N}{\lambda} \mathrm{Tr}(\cdots)$$

$$\Rightarrow \quad \log Z = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \qquad \boxed{\text{Topological}}$$

- Why orientable surface? because $\rightleftarrows$, $SU(N)$ is hermitian matrix, lower and up indices are in different foots. So if we consider $SO(N)$, we need to use unorientable surface to describe its t'Hooft limit.

- observerbles in gauge theory are more limited, operator don't must need to preserve global symmetry. but they must need to invariant under gauge symmetry. So the observables are:

$$\mathrm{Tr}(\Phi^2),\ \mathrm{Tr}\, F^{\mu\nu} F_{\mu\nu},\ \cdots \qquad (\text{single-trace},\ \mathcal{O}_n)$$

so multi-trace operator can be denoted by

$$:\mathcal{O}_1 \mathcal{O}_2:\ ,\ :\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3:\ ,\ \cdots \quad \text{such as}$$

$$\mathrm{Tr}(\Phi^2)\, \mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) \ \cdots$$

As before. we want to expand $\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle_c$ by $1/N$. not only satisfied by $\langle 0 \rangle_c = \log Z$.

Consider $Z[J_1 \cdots J_n] = \int \mathcal{D}A\,\mathcal{D}\Phi \exp\left[i S_0 + i\int J_i(x) O_i(x)\right]$

is the generator.

$$\langle O_1 \cdots O_n \rangle_c = \frac{1}{i^n} \frac{\delta^n \log Z}{\delta J_1(x_1) \cdots \delta J_n(x_n)}$$

☆ missing part in textbooks

Trick $\quad Z \longleftrightarrow \int \mathcal{D}A\,\mathcal{D}\Phi \exp[\underbrace{i S_0 + iN \int J_i O_i dx}_{i S_{eff}}]$

$$\langle O_1 \cdots O_n \rangle_c = \frac{1}{i^n} \frac{\delta^n \log Z}{\delta J_1(x_1) \cdots \delta J_n(x_n)} \cdot \frac{1}{N^n}$$

because $S_0 \sim N \,\mathrm{Tr}[\cdots]$ so $S_{eff} \sim N\,\mathrm{Tr}[\cdots]$ vacuum-like.

$$\Rightarrow \log Z[J_1 \cdots J_n] = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda)$$

$$\Rightarrow \langle O_1 \cdots O_n \rangle_c \sim \sum_{g=0}^{\infty} N^{2-2g-n} f_n^{(g)} \sim N^{2-n} \left(\underset{\text{planar}}{1} + \underset{\text{non-planar}}{\mathcal{O}\left(\tfrac{1}{N^2}\right)} \right)$$

Physics interpretation.

$\langle O_i O_j \rangle_c \sim \mathcal{O}(1) \xrightarrow{\text{diagonalize}} \langle O_i O_j \rangle \sim \delta_{ij}$ diagonalized

$\langle O_i : O_1 O_2 : \rangle_c \sim \mathcal{O}\left(\frac{1}{N}\right) \xrightarrow{N\to\infty} 0$

$\langle :O_1 O_2: \; :O_1 O_2: \rangle \overset{\text{wick}}{=} \underbrace{\langle :O_1 O_2: \; :O_1 O_2: \rangle_c}_{\mathcal{O}\left(\frac{1}{N^2}\right) = 0}$

$\qquad + \langle O_1 O_1 \rangle_c \langle O_2 O_2 \rangle_c$

connected and disconnected

So we can interpret $O_i |0\rangle$ as a single particle

In QCD we call them glueball states

$:O_i O_j: |0\rangle$ as a 2-particle

$:O_1 \cdots O_n: |0\rangle$ as a n-particle fluctuations of "glue balls" suppressed. suppose $\langle O_i \rangle \neq 0$

$$\sigma_{O_i} = \langle O_i^2 \rangle - \langle O_i \rangle^2 = \langle O_i^2 \rangle_c \sim O(N^0)$$

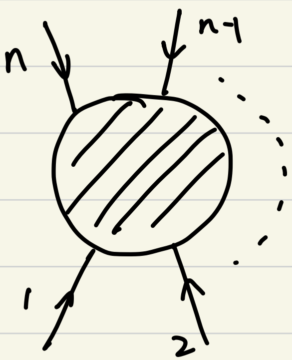
But $\langle O_i \rangle \sim O(N^1)$ $\quad (\langle O_i \rangle_c = \langle O_i \rangle)$

so $\dfrac{\sqrt{\sigma_{O_i}}}{\langle O_i \rangle} \sim \dfrac{1}{N} \longrightarrow 0$

and $\langle O_1 O_2 \rangle = \underbrace{\langle O_1 \rangle \langle O_2 \rangle}_{\sim O(N^2)} + \underbrace{\langle O_1 O_2 \rangle_c}_{\sim O(N^0)}$

$\overset{N\to\infty}{\rightsquigarrow} \langle O_1 \rangle \langle O_2 \rangle$

so in Large N limit there is no Quantum fluctuation. It's just a classical theory

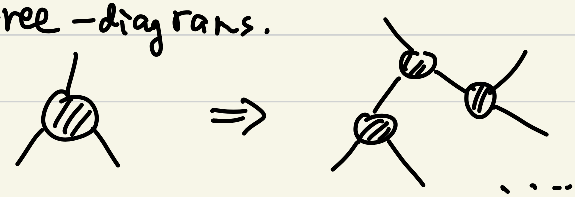


$\sim \langle O_1(x_1) \dots O_n(x_n) \rangle_c$

as "scattering amplitudes" of n "glue balls", then to leading order in $N \to \infty$. the Scatterings are classical.

Meanings only involve tree lever scatterings

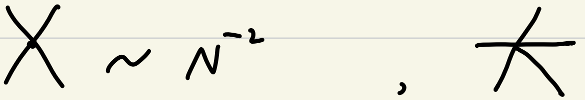
like BCFW, we can use three-vertex to obtane tree-diagrams.



3-vertex "coupling constant" is $1/N$. than A_n involves $n-2$ such 3-vertices. $\Rightarrow A_n \sim N^{2-n}$

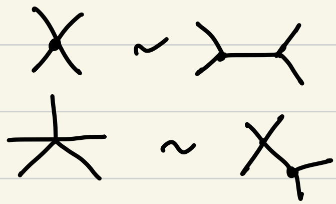
this is exactly we have obtained before.

you can also include higher order vertices like



$\sim N^{-2}$, $\sim N^{-3}$.

But include them doesn't change the large-N conclusion. because they can be degenerate to 3-vertex.



$\sim N^{-1} \cdot N^{-1} \sim N^{-2}$

$\sim N^{-2} \cdot N^{-1} \sim N^{-3}$

so the underlying is that $\langle O_1 \cdots O_n \rangle \sim N^{2-n}$

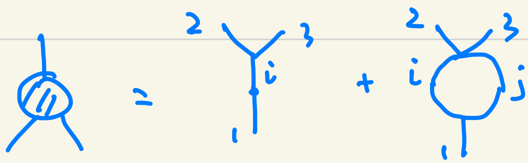
About intermediate states: To leading order in N. in any Correlation functions. there are only. one-particle intermediate state.

e.g. $$\langle O_1 O_2 O_3 \rangle = \sum_i \underbrace{\langle O_1 O_i \rangle}_{\sim O(1)} \underbrace{\langle O_i O_2 O_3 \rangle}_{\sim O(N^{-1})}$$

$$+ \sum_{ij} \underbrace{\langle O_1 :O_i O_j: \rangle}_{\sim O(N^{-1})} \underbrace{\langle :O_i O_j: O_2 O_3 \rangle}_{\sim O(N^{-2})}$$

$$+ \cdots$$

$\Rightarrow$ Leading order $O(\frac{1}{N})$ ← one-particle indetermediate state's Contribution.



so, actually, it is the collary of any loop diagrams are suppressed

All in all. $SU(\infty)$ is a Classical theory at leading $\frac{1}{N}$-order. A classical theory of glue-balls.

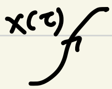
gauge theory = glue ball theory

$N \to \infty$, $\hbar \sim O(1)$ $\qquad$ $\hbar \to 0$

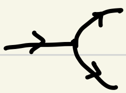
Large N expansion $\longleftrightarrow$ semi-classical expansion $\hbar \sim \frac{1}{N}$

Gauge/String Duality.

QFT: Wordline formalism, the first Qantization approach



$\longrightarrow \int \mathcal{D}X^\mu \, e^{iS_{particle}}$, $S_{particle} = m\int dl$



gives you the interactions. need to add by hand

*ps: second Quantization approach. we don't quantize the quantum motion $x^\mu(\tau)$, but denote them as parameters of Quantum field. eg. $A^\mu(x)$.

String: Generalization of this first Quantization approach

$$S_{string} = \underset{\frac{1}{2\pi\alpha'}}{\underset{\uparrow}{T}} \int_\Sigma dA_{World\text{-}sheet} \qquad \text{(Nambu-Goto)}$$

↙ Vacuum energy

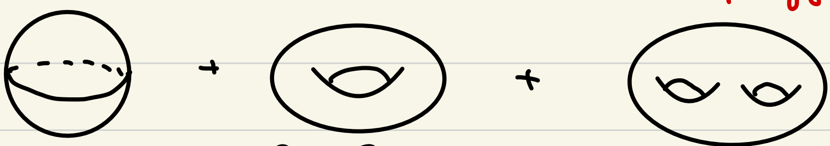
$$Z_{\text{string}} = \sum_{\substack{\text{all closed} \\ \text{2d surfaces}}} e^{-S_{NG}}$$

↖ Euclidean signature

* Consider closed string
later we will consider
open string

$$= \sum_{h=0}^{\infty} e^{-\lambda \chi} \sum_{\substack{\text{surface} \\ \text{with genus } h.}} e^{-S_{NG}} \qquad g_s := e^{-\lambda}$$

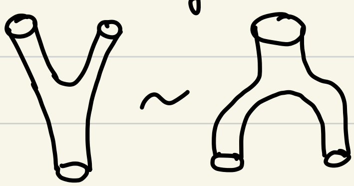
↑ Weight for different topology.



$\mathcal{O}(g_s^{-2})$ + $\mathcal{O}(g_s^{0})$ + $\mathcal{O}(g_s^{2})$ + ...

Summing over topology automatically includes interactions of strings, we don't need to add by hand.

Basic string vertices (pants decomposition)



$\sim g_s^{+1} \sim e^{-\lambda}$: string coupling

Now include external strings.

$\chi = 2-2h + b$, b is the # of boundaries.(external strings)

$$A_n^{\text{string}} = \sum_{h=0} g_s^{n-2+2h} F_n^{(h)}$$

identical mathematical structure with large N expansion

$e^{\lambda} = g_s \longleftrightarrow \frac{1}{N}$

external strings $\longleftrightarrow$ glue-ball single-trace operators

Sum over world-sheet topology $\longleftrightarrow$ sum over double-line Feynman diagram

$$f_n^{(h)} = \sum_{\substack{\text{all feynman-diagrams} \\ \text{with genus } h}} A = \sum_{\substack{\text{sum over all} \\ \text{possible triangulations} \\ \text{of a surface}}} A$$

$\Uparrow$

$$\sum_{\substack{\text{Sum over all} \\ \text{2d surfaces with} \\ \text{genus } h}} A \Leftarrow$$ all 2d surfaces can be triangulized.

the key is "Feynman diagram is a partition of 2d surface" what we said before.

Geometric picture $\Downarrow$

$$F_n^{(h)} = \sum_{\substack{\text{Sum over all} \\ \text{2d surfaces with} \\ \text{genus } h}} \int \mathcal{D}X\, e^{-S_{NG}}.$$

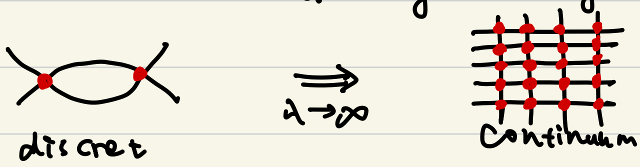
Can be gauge fixing to $\int \mathcal{D}X\, \delta(g-\hat{g})$ by F-P ghost.

$\simeq \int \mathcal{D} g_{ab} / \text{diff} \times \text{weyl}$

All the above clues tell us the large N gauge theory is the string theory, glue-ball states behave like string states. Single-trace operator as vertex operator

But how to connect Feynman diagram amplitudes with $e^{-S_{string}}$? Which type of string theory does it correspond to

Because String theory is a Continuum theory. But Large N theory is related to triangularations of Surfaces. so it's a discret theory, so we expect the Geometric picture of G only manifest in strong coupling limit, i.e. $\lambda \to \infty$ or Consider Feynman diagrams with many Vertices

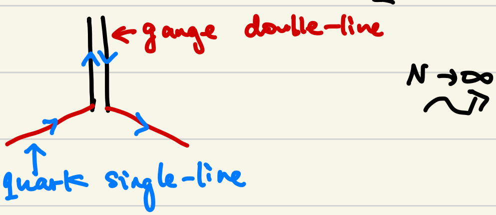


recall $A_{YM} \sim \lambda^{L-1} N^{\chi}$

so $\lambda \to \infty$ means diagram with so many Loops (vertices) dominant

Actually, for large N theory of Random Matrix Theory (Quantum Mechanics version of YM theory). One could go pretty far in relating them to some low-dimensional String theory. (hep-th/9108019)

Generalization with quarks.



Vacuum Feynman diagram is Corresponding to 2d surfaces With boundary ← quark loops

$\updownarrow \frac{1}{N} \sim g_s$

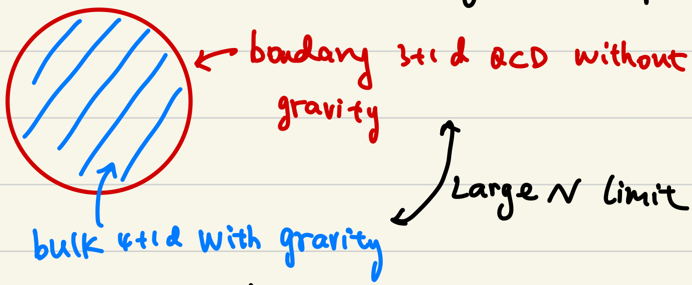
string theory include both open string and closed string

Generalizations of gauge groups.

gauge group in Large N $\longleftrightarrow$ chan-paton in string Theory

$SO(N)$, $Sp(N)$ $\Rightarrow$ Unoriented. string theory

QCD doesn't include gravity, but string theory does.
so if we want to connect a non-gravity theory to a gravity theory. we need to use holographic principle



To make a consistent string theory in $D \neq 10, 26$. we need to discard the weyl gauge symmetry. Then we don't need $D=10, 26$ to cancel the weyl (a.k.a. conformal) anomaly. This is realized by Liouville string theory

Now, suppose we consider CFT, not just QCD then

$$\text{Large } N \text{ CFT in } Mink_d \xleftrightarrow{\frac{1}{N} \sim g_s} \text{String theory in } AdS_{d+1}$$

Why AdS? String theory should also conserve the d dim.

poincaré symmetry. so $Mink_d$ should be a sub-manifold

$$ds^2 = a(z)^2 \left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right)$$

CFT has scale invariance : $x^\mu \mapsto \lambda x^\mu$

d+1 dim string should also preserve it, so $a(\lambda z) \sim \frac{1}{\lambda} a(z)$

$$\Rightarrow \quad a(z)^2 = \frac{R^2}{z^2}$$

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right)$$

this space-time is AdS_{d+1}.

chronicle

- 1974: (Wilson) lattice QCD
- 1993(4): ('t Hooft, Susskind) Holographic principle
- 1995: (polchinski) D-brane
- 1997 June: need 5d string theory to describe QCD (Polyakov)
- 1997 Nov: (Maldacena) AdS/CFT
- 1998: connection between gauge/string duality and Holographic principle (Witten)

String and D-brane

$$Z = \int \mathcal{D}\gamma_{ab} \mathcal{D}X^\mu \, e^{iS_p[\gamma, X]}$$ ($\frac{1}{\text{diff}\times\text{Weyl}}$ has been omitted)

we don't fix gauge now

$$S_p[\gamma, X] = -\frac{1}{4\pi\alpha'}\int d\sigma^2 \sqrt{-\gamma}\, \gamma^{ab}(\tau,\sigma)\, \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)$$

String theory can be considered a 2d "gravity" coupled to D free scalar fields

on the worldsheet

Consider $g_{\mu\nu} = \eta_{\mu\nu}$. i.e the background is Mink_D.

Symmetry: global: poincáre (target space time)

(first class) gauge: diff $\times$ weyl (world sheet space time)

$\Rightarrow$ another term consistent with these symmetry

$$S_{\text{euler}} = \frac{\lambda}{4\pi}\int d^2\sigma \underbrace{\sqrt{-\gamma} R}_{}$$ $\lambda := \langle\phi\rangle$ a scalar.

total derivative

But "total derivative" is a local description, actually it has global topological effect

In 2D: $\chi = \frac{1}{4\pi}\int d^2\sigma \sqrt{\gamma} R = 2 - 2\cdot \text{genus}$.

In Euclidean path integral. $e^{-S_{\text{euler}}} = e^{-\lambda\chi}$

String spectrum.

Light-cone Quantization

We will use the constraints to fix gauge first, and then quantization

$$\frac{\delta S}{\delta \gamma_{ab}} = 0 \Rightarrow T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X_\mu = 0$$

$$\frac{\delta S}{\delta X^\mu} = 0 \Rightarrow \partial_a(\sqrt{-\gamma}\,\gamma^{ab}\partial_b X^\mu) = 0$$

boundary condition for open string

$$\delta X^\mu \gamma^{\sigma b}\partial_b X_\mu\big|_{\sigma=0,\pi} = 0$$

Dirichlet: $\delta X^\mu\big|_{\sigma=0,\pi} = 0$

Neuman : $\partial^\sigma X_\mu\big|_{\sigma=0,\pi} = 0$

Fix diff × weyl gauge symmetry. Using isothermal coordinate : $\gamma_{ab} = \eta_{ab}$

E.o.M of X^μ becomes : $\partial_\sigma^2 X^\mu - \partial_\tau^2 X^\mu = 0$

...... γ_{ab} becomes : $T_{00} = T_{11} = \frac{1}{2}\left[(\partial_\tau X)^2 + (\partial_\sigma X)^2\right] = 0$

Virasoro Constraints $\longrightarrow$ $T_{01} = T_{10} = \partial_\tau X \cdot \partial_\sigma X = 0$

boundary condition : $\gamma^{\sigma b}\partial_b X^\mu = 0 \Leftrightarrow \partial_\sigma X^\mu = 0$

solution of e.o.m: (ignore the Virasoro constraints)

$$X^\mu(\sigma,\tau) = x^\mu + v^\mu \tau + X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma)$$

closed string. X_L, X_R are independent function

open string (Neuman) : $X_L'(\tau) = X_R'(\tau)$, $X_L'(\tau-\pi) = X_R'(\tau-\pi)$

$$\Rightarrow X_L = X_R \text{ is periodic in } 2\pi$$

After fix $\gamma_{ab} = \eta_{ab}$, there are still other residual gauge freedom. Introduce $\sigma^\pm := \frac{1}{\sqrt{2}}(\tau \pm \sigma)$

Worldsheet Metric : $ds^2 = -d\tau^2 + d\sigma^2 = -2d\sigma^+ d\sigma^-$, this metric is preserved by : $\sigma^\pm \to f^\pm(\sigma^\pm)$ - diff

$$ds^2 \to \tilde{ds}^2 = -2f^{+\prime}(\sigma^+) f^{-\prime}(\sigma^-) d\sigma^+ d\sigma^-$$ - weyl

$\tilde{ds}^2 \overset{\text{Weyl}}{\sim} ds^2$. So it's conformal invariant.

Under this transformation

$$\tilde{\tau} = \frac{f^+(\tau+\sigma) + g^+(\tau-\sigma)}{\sqrt{2}}$$

obviously $\partial^2 \tilde{\tau} = 0$, has the same form as the classical solution of X^μ. so we can choose

$$\tau = \frac{X^+}{v^+} \quad , \quad X^\pm := \frac{X^0 \pm X^1}{\sqrt{2}}$$

if $\partial^2 \tilde{\tau} \neq 0$ we can not do that, but it is.

to fix the gauge, which called light-cone gauge

Virasoro constraints become :

$$2v^+ \partial_\tau X^- = (\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2 \qquad i \neq 0,1$$

$$v^+ \partial_\sigma X^- = \partial_\tau X^i \partial_\sigma X^i$$

So X^- can be fully solved in terms of $X^i(\sigma,\tau)$

As we said before X^+ is fully fixed by $X^+ = v^+\tau$

so the independent X^μ are X^i We cut off 2 d.o.f , we only need to consider D-2 free

$X^+ = p^+\tau$ has no oscillating terms, only a zero mode.

Scalar fields, and because $(\partial X^0)^2$ has wrong kinematic term sign, naively we think this system will be unstable. but now we can omit X^0 by using constraints and gauge fixing. Then ligh-cone gauge (partly) solve the problem of unitary at quantum level. But in light-cone gauge. Lorentz symmetr is not manifest, remaining $SO(D-2)$ symmetry.

Mode expansion.

$$X_L^\mu(\sigma+\tau) = i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0}^{+\infty} \frac{\alpha_n^\mu}{n} e^{-in(\tau+\sigma)}$$

$$X_R^\mu(\tau-\sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0}^{\infty} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau-\sigma)}$$

$$\alpha_n^\mu = \bar{\alpha}_{-n}^\mu$$
$$\tilde{\alpha}_n^\mu = \bar{\tilde{\alpha}}_{-n}^\mu$$

for open string, $\alpha_n^\mu = \tilde{\alpha}_n^\mu$, Left and right mode are not independent. Center of Mass Motion of the String is given by the zero mode:

$$X^\mu + v^\mu \tau = \frac{1}{2\pi}\int_0^{2\pi} d\sigma X^\mu(\sigma,\tau)$$

Using $2v^+\partial_\tau X^- = (\partial_\tau X^i)^2 + (\partial_\sigma X^i)^2$ and $v^+\partial_\sigma X^- = \partial_\tau X^i \partial_\sigma X^i$ to solve the mode of X^-: v^-, α_n^- and $\tilde{\alpha}_n^-$

For closed string:

$$2v^+v^- = v_i^2 + \alpha' \sum_{m\neq 0} (\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i)$$

$$\sum_{n\neq 0} \alpha_{-n}^i \alpha_n^i = \sum_{n\neq 0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \quad \text{(level matching condition)}$$

for open string:

$$2v^+v^- = v_i^2 + 2\alpha' \sum_{m\neq 0} \alpha_{-m}^i \alpha_m^i$$

And no level matching condition.

Translation symmetry Noether current $j_a^\mu = \frac{1}{2\pi\alpha'} \partial_a X^\mu$

$\Rightarrow$ $p^\mu = \int_0^{2\pi} d\sigma\, j_\tau^\mu$ is the conserve current, momentum

For open string: $v^\mu = 2\alpha' p^\mu$

$$\underbrace{2p^+p^- - p_i^2}_{-p^2 = M^2} = \frac{1}{2\alpha'} \sum_{m\neq 0} \alpha_{-m}^i \alpha_m^i$$

$$\Rightarrow M^2_{open} = \frac{1}{2\alpha'} \sum_{m\neq 0} \alpha_{-m}^i \alpha_m^i$$

For closed string: $v^\mu = \alpha' p^\mu$

$$\Rightarrow M^2_{closed} = \frac{1}{\alpha'} \sum_{m\neq 0} \left[\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i\right]$$

Now, quantize α^i (i.e. X^i), the canonical world sheet momentum density is $\Pi^i \equiv j_\tau^i = \frac{1}{2\pi\alpha'} \partial_\tau X^i$, impose canonical quantization condition

$$[X^i(\sigma,\tau), X^j(\sigma',\tau)] = 0, \quad [\Pi^i(\sigma,\tau), \Pi^j(\sigma',\tau)] = 0$$

$$[X^i(\sigma,\tau), \Pi^j(\sigma',\tau)] = i\delta^{ij}\delta(\sigma-\sigma')$$

$\Rightarrow [x^i, p^j] = i\delta^{ij}$,

$[\alpha_m^i, \alpha_n^j] = [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta^{ij}\delta_{m,-n}$

Creation operator : $\frac{1}{\sqrt{m}}\alpha_m^i$

annihilation operator : $\frac{1}{\sqrt{m}}\alpha_{-m}^i$

$\Bigg\}\ m > 0$

Spectrum :

<u>$N=0$</u> vacuum

$$\alpha_m^i|0,p\rangle = \tilde{\alpha}_m^i|0,p\rangle = 0 \quad \text{for all } i, m.$$

<u>$N>0$</u> excited states

$$\alpha_{-m_1}^{i_1}\alpha_{-m_2}^{i_2}\cdots\alpha_{-n_1}^{j_1}\alpha_{-n_2}^{j_2}\cdots|0,p\rangle$$

(openstring only α_{-n}^i)

Oscillator number : $\alpha_{-m}^i\alpha_m^i = mN_m^i$, no summation.

Mass-shell conditions:

$$M_{open}^2 = \frac{1}{\alpha'}\sum_{i=2}^{D-1}\sum_{m=1}^{\infty} mN_m^i + a_0$$

a_0 is the summation of zero point energy of oscillators. it comes from $[a_m, a_n] \neq 0$, each oscillator contributes $\frac{1}{2}\omega$, ω is the frequency In present, $\omega = m$. so :

$$a_0 = \frac{1}{\alpha'}\sum_{i=2}^{D-1}\sum_{m=1}^{\infty}\frac{1}{2}\cdot m$$

For closed string :

$$M^2_{closed} = \frac{2}{\alpha'} \sum_{i=2}^{D-2} \sum_{m=1}^{\infty} (m N_m^i + m \tilde{N}_m^i) + a_0'$$

$$a_0' = \frac{2}{\alpha'} \sum_{i=2}^{D-2} \sum_{m=1}^{\infty} (\underset{\text{Left}}{\frac{1}{2}} + \underset{\text{Right}}{\frac{1}{2}}) \cdot m \,.$$

Riemann-zeta Regularization, $\zeta(-1) = -\frac{1}{12}$

$\Rightarrow a_{open} = -\frac{D-2}{24}\frac{1}{\alpha'}$, $a_{close} = -\frac{D-2}{24}\frac{4}{\alpha'}$

open string spectrum.

T. Tachyon : $|0, p^\mu\rangle$, $M^2 = -\frac{D-2}{24}\frac{1}{\alpha'}$

A_μ Vector Bosons: $\alpha_{-1}^i |0, p^\mu\rangle$, $M^2 = \frac{1}{\alpha'}(1 - \frac{D-2}{24})$

In light-cone gauge, Lorenz Symmetry is hidden. To recover it, $SO(D-2)$ Vector Bosons should be massless

$$\boxed{D = 26}$$

Otherwise, the Lorenz symmetry will be lost in quantum level

Higher excitations: all massive, with spacing by $\frac{1}{\alpha'}$:

$\alpha_{-1}^i \alpha_{-1}^i |0, p^\mu\rangle$, $\alpha_{-2}^i |0, p^\mu\rangle$ Regge slope

closed string spectrum

T. Tachyon : $|0, p^\mu\rangle$, $M^2 = -\frac{4}{\alpha'}\frac{D-2}{24}$

$g^{\mu\nu}$ ϕ $B^{\mu\nu}$

Graviton ⊕ Dilaton ⊕ B-field : $\alpha_{-1}^i \tilde{\alpha}_{-1}^i |0, p^\mu\rangle$, $M^2 = \frac{26-D}{6\alpha'}$

Again, closed string can only live in D=26 likes open string.

Higher modes are all Massive.

Low Energy EFT:

$S_{LEE} = -\frac{1}{16\pi G_N}\int d^{26}x\sqrt{-g}\,e^{-2\phi}\left[R-4(\partial\phi)^2+\frac{1}{12}H^2\right]$, $H_{\mu\nu\lambda}=\partial_{[\mu}B_{\nu\lambda]}$

Einstein Gravity: $M_4 \propto G_N$. $M_4^{cl} \propto g_s^2 \Rightarrow \boxed{G_N \propto g_s^2}$

But this theory is non-renormalizable, In string theory, all loop diagrams are finite.

$\Rightarrow S_{LEE} \sim \frac{1}{g_s^2}\int e^{-2\Phi}R \Rightarrow \boxed{g_s = e^{\langle\Phi\rangle}}$ Φ is moduli prameter

Everything is in 「some sense」 determined!

moduli stabilization Problem

Superstring (world sheet susy. RNS formalism)

$$S_{super} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_a X^\mu\partial^a X_\mu + i\bar{\psi}\gamma^a\partial_a\psi\right)$$

one loop unitary $\Rightarrow$ GSO projection $\Rightarrow$ Quantization without tychn

type II A: $h_{\mu\nu}$ $B_{\mu\nu}$, ϕ, A_μ, $C_{\mu\nu\lambda}$ + fermions

II B: ...., χ, $C_{\mu\nu}$, $C_{\mu\nu\lambda\rho}$ + fermions

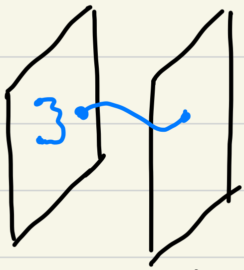
R-R forms and their difference in A\B theory are very important for disscusion on D-brane

Strength of open string interaction.

$$\boxed{g_s \sim g_o^2}$$

D-brane

D is for Dirichlet boundary condition.



For Dirichlet BC:
end points of open string should on D-brans

$SO(9,1) \ltimes \mathbb{R}^{9,1}$ symmetries are broken down by D-brane

$$SO^{+}(9,1) \ltimes \mathbb{R}^{9,1} \longrightarrow \mathbb{R}^{p} \times SO^{+}(1,p) \times SO(D-1-p)$$

Dp branes should be Considered as non-perterbative topological defect in string theory.

Note : X^0 Cannot have Dirichlet BC. But for Euclidean Spacetime, we can even impose Dirichlet BC on X^0. We obtain D-1 brane

String can also end on D-brans with different dimentions

Open string Spectrum on Dp-brans.

$\mu = (\alpha, a)$ $\alpha = 0, 1, \dots p$ (N) , $a = p+1, \dots, D-1$ (D)

As before: $X^\alpha(\sigma,\tau) = x^\alpha + 2\alpha' p^\alpha \tau + \sum_{n\neq 0} \frac{\alpha_n^\alpha}{n} e^{-in\tau} \cos n\sigma$

For the Dirichlet BC the most general solution is:

$$X^a(\sigma,\tau) = x^a + 2\alpha' p^a \tau + X_R^a(\tau-\sigma) + X_L^a(\tau+\sigma)$$

Using the BC: $X^a(0,\tau) = X^a(\pi,\tau) = b^a$

$$\Rightarrow \quad X^a(\sigma,\tau) = b^a + \sum_{n\neq 0} \frac{\alpha_n^a}{n} e^{-in\tau} \sin n\sigma$$

The only difference between N and D BC is that $p^a = 0$, but the mass shell condition unchange.

excitation states are still:

$$\alpha_{-n_1}^{\mu_1} \alpha_{-n_2}^{\mu_2} \cdots \alpha_{-n_i}^{\mu_i} |0; p^a\rangle, \quad M^2 = \frac{1}{\alpha'}(N-1) + \frac{1}{2\pi\alpha'} \sum_a (b_1^a - b_2^a)^2$$

★ But they are particles in $(p+1)$-dimensional world volume of D-branes, and fall into represents of

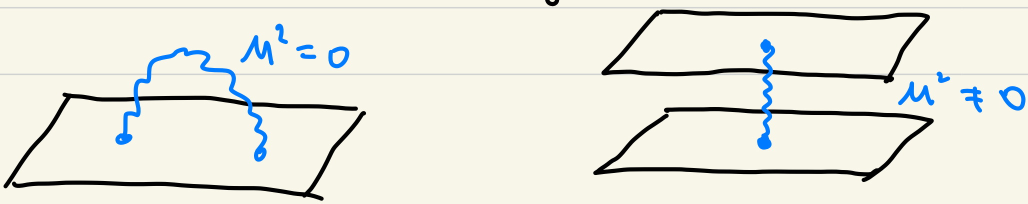
$$(SO^+(1,p) \ltimes T^p) \times SO(D-p-1)$$

In light-cone gauge, $\alpha = +, -, i$, the massless states are

$$\alpha_{-1}^i |0, p^\alpha\rangle, \quad \alpha_{-1}^a |0, p^\alpha\rangle$$

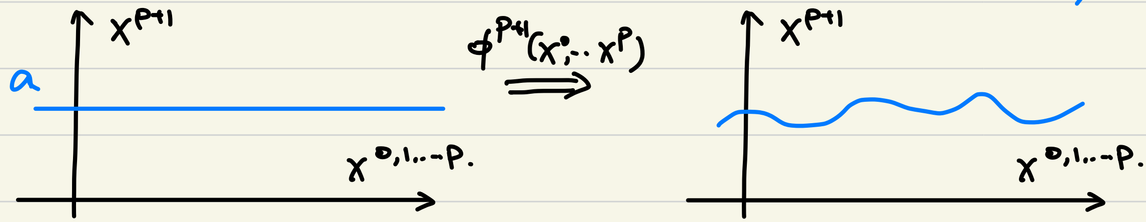
Respect to world volume poincaré group, the former is vector field, the latter is scalar field

And two endpoints of string should end on same D-brane



d.o.f of scalar fields = transverse directions.

They can be interpreted as fluctuations of the D-brane in transverse directions (collective coordinates)



Curved D-brane means non-trivial scalar field excitations. so at world volume level. having ϕ^a excitation modifies the Dirichlet B.C. So D-brane should be consider as a fundamental object in string theory, not just as a rigid B.C. This can also be seen by low energy EFT.

(chern-simons term should be added)

$$S_{DBI} = -T_p \int d^{p+1}x \sqrt{-|G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}|}\ , \quad G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$$

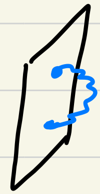
$$X^a = b^a + \Phi^a \Rightarrow S_{LEE} = -T_p \int d^{p+1}x \left(1 + \frac{1}{4} F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\partial^\alpha \Phi^a \partial_\alpha \Phi^a + \dots\right)$$

$$\Phi^a = \Phi^a(t) \Rightarrow S_{LEE} \sim -\int dt\, M_{Dp}\left(1 + \frac{1}{2}\dot{\Phi}^a + \dots\right), \quad M_{Dp} := T_p V_p$$

so Φ^a can be interpreted as center of mass position of D-brane when there is no fluctuation. D-bran has life!

Tension of a D-brane

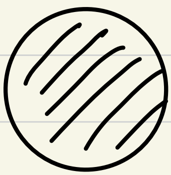
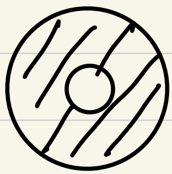
mass of a D-brane = Vacuum energy of open strings living on it



$M_{Dp} = T_p V_p = E_{vacuum}$

$= \sum$ Vacuum diagram of open strings.

$= \sum$ all 2-dim surfaces with at least one boundaries, but no external open strings

$=$  $+$  $+ \cdots$

disK anulus

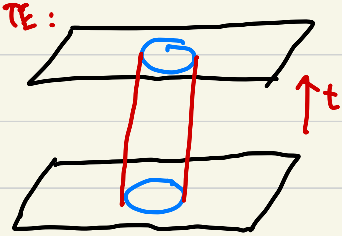
NOTE: There is no vertex operator on the boundaries

For weak coupling $g_s \ll 1$

$$\boxed{T_p = \frac{1}{g_s (2\pi)^p (\alpha')^{(p+1)/2}}}$$

which can be derived from type IIA $\xleftrightarrow{\text{T-duality}}$ IIB

NOTE:

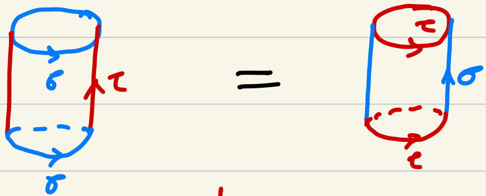


The Leading interaction between two D-brane. associate to Graviton and dilaton.

this cylinder topology gives g_s^0 order and T_p

comes from D-brane mass, so the gravitational force between two D-branes $\sim G_N T_p^2$, we said before $G_N \sim g_s^2$, so this diagram tells us $\boxed{T_p \sim \frac{1}{g_s}}$ the proportional coefficient is derived by using string duality.

Alternatively, you can also consider the cylinder as an open string diagram



Gravity
Tree leve in closed String
||
1-loop level in open string
Gauge Theory

Channel Duality.

Remark:

Q: Why modes describing motions of D-branes appear as massless modes?

A: D-brans break the translation symmetry. ϕ^a is the Goldstone bosons which are massless.

Multiple coincidental D-branes.

there are 4 type open string ending on D-brans

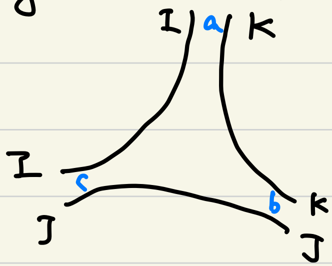
$1 \to 2$, $1-1$, $2 \to 1$, $2-2$

They have indentical excitations

$|\psi, IJ\rangle$ $I, J = 1, 2$, so we get 4 copies of massless spectrum. Each open string excitation becomes a 2×2 matrix. e.g. $(A_\alpha)^I{}_J$, $(\phi^a)^I{}_J$

It can be generalized immediately to n branes

Strings should interact by joining their ends.



$\sim T_a^{IK} T_b^{KJ} T_c^{JI}$

Symmetry Enhancement : $\underbrace{U(1) \times \cdots \times U(1)}_{N} \longrightarrow U(N)$

On the world sheet $U(N)$ is a global symmetry, but in spacetime, it's a gauge symmetry, $(A_\alpha)^I{}_J$ must be the corresponding gauge bosons. And the low energy EFT is Yang-Mills-Scalar theory.

$$\boxed{S_{YMS} = -\frac{1}{g_{YM}^2}\int d^{p+1}x \, \mathrm{Tr}\left\{\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}(D_a\phi^a)^2 + [\phi^a, \phi^b]^2\right\}}$$

$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta]$, $D_a\phi^a = \partial_a\phi - i[A_a, \phi^a]$

$g_{YM}^2 = g_s \cdot \alpha'^{\frac{p-3}{2}} \cdot c_p$, where c_p is a numerical coefficient

Recall that when we consider excitation between two separately D-branes, A_α and ϕ^a will not be massless

$$\boxed{M \sim \frac{d}{2\pi\alpha'} \sim d\, T_{DP}}$$

d is the distance between D-branes. Massless to massive modes means that gauge symmetry S.B. by separating D-branes

Details about the Higgs mechanism of separating D-branes:

We will only consider $U(2) \to U(1) \times U(1)$ here, any further details can be found in Zwiebach §14.3 or Polchinski Vol. I §8.

The effective low energy YMs Lagrangian is

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} D_\mu \phi\, D^\mu \phi\right)$$

where $A_\mu = \frac{1}{2}(A_\mu^0 \mathbb{1} + A_\mu^i \sigma_i)$, $D_\mu \phi = \partial_\mu \phi - i[A_\mu, \phi]$

Now study the spectrum in the Higgs phase where the Scalar ϕ has expectation value

$$\phi_0 = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} , \quad v = \frac{d}{2\pi\alpha'}$$

mass calculated by string theory.

$$\Rightarrow \frac{1}{2}(D_\mu \phi_0)(D^\mu \phi_0) = \frac{v^2}{2}\left((A^1)^2 + (A^2)^2\right) .$$

Thus two of the gauge fields have received mass v.

Then $U(2)$ breaks down to $U(1) \times U(1)$.

D-brane in Superstring

SUST excludes tachyon from spectrum and D-brane will carry R-R charges, so they are stable object.

type IIA : $C^{(1)}_{\mu}$, $C^{(3)}_{\mu\nu\rho}$

type IIB : χ , $C^{(2)}_{\mu\nu}$, $C^{(4)}_{\mu\nu\rho\lambda}$ ← self-dual

p-form can couple with p-dim world volume which can be swept by D_{p-1} branes. So Dp brane can couple with $C^{(p+1)}$ and the gauge symmetry on $C^{(p+1)}$ tells us Dp branes carry conserved R-R charge

Additionally, you can generalize magnetic charge in EM to R-R fields. So each R-R "electrical" charge corresponds to its magnetical dual magnetic charge.

$$\boxed{d\tilde{C}^{(D-n-2)} = * dC^{(n)}}$$

So $C^{(n)}$ can also magnetically couple with $D-n-3$ brane

		electric	magnetic
IIA	$C^{(1)}_{\mu}$	D0	D6
	$C^{(3)}_{\mu\nu\rho}$	D2	D4
IIB	$C^{(2)}_{\mu\nu}$	D1 (D-string)	D5
	$C^{(4)}_{\mu\nu\rho\lambda}$	D3	D3

self-dual

so χ will **not** be cosidered

D-instanton

χ D-1 D7

give rise to a deficit angle in the geometry.

On these stable branes, SYM theory lives on particularly, $\mathcal{N}=4$, $D=4$ SYM lives on D3 brane

D-branes as spacetime geometries

consider e.g. a charged particle sitting at $\vec{r}=0$ of 4d spacetime, the Largrangian,

$$\mathcal{L} = \frac{1}{16\pi G_4} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

EOM: $\quad \partial_\mu F^{\mu\nu} = j^\nu \quad , R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_4 T_{\mu\nu}$

$$j^0 = q\,\delta(\vec{r}) \ , T^{\mu\nu} = T_p^{\mu\nu} + T_{em}^{\mu\nu}$$

where the particle Energy-Momentum tensor is $T_p^{00} = m\delta(\vec{r})$ other Components are 0.

These formulas tell you how a charged particle deforms the spacetime background.

$\Rightarrow A_0 = \frac{q}{4\pi r}$ (i.e. $\int_{S^2} {*F} = q$), $\quad ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_2^2$

Where, $\Delta = 1 - \frac{2MG_4}{r} + \frac{q^2 G_4}{r^2}$, it's Reissner-Nordström solution

Similarly, for magnetic charged particle. $\int_{S^2} F = g$. Dirac Quantization tells us $q \cdot g = 2\pi \mathbb{Z}$. The solution of Einstein Field Eq is RN solution again.

Now, generalize from this D0 example to Dp branes e.g. D3 brane in type IIB, but we will consider the low energy EFT — type IIB Supergravity

$$\boxed{16\pi G_N = (2\pi)^7 g_s^2 \alpha'^4}$$

regime of validity:

① $g_s \ll 1$. quantum corrections for classical gravity is small

② energy scale$^2 < \frac{1}{\alpha'}$, Massive modes can be ignored

③ Curvature $\ll \frac{1}{\alpha'}$, string can be approximated by point particle

$$\boxed{\alpha' \to 0 \, , \; g_s \to 0}$$

D3 branes carry both electric and magnetic charges.

$$g_3 = \int_{S^5} * F^{(5)} \overset{\text{self-dual}}{=\!=\!=} \int_{S^5} F^{(5)} = g_3$$

$$g_3 g_3 = 2\pi \mathbb{Z} \Rightarrow g_3^2 = g_3^2 = 2\pi \mathbb{Z}$$

For N D3-branes. $g_3 = g_3 = \sqrt{2\pi} N$ and the tension of D3 is

$$\boxed{T_3 = \frac{N}{(2\pi)^3 g_s \alpha'^2} = \frac{g_3}{\sqrt{16\pi G_N}}}$$

mass - charge equivalence, so D3-brane considered here is BPS state, which is supersymmetric object.

Solve the EOMs as before:

$$ds^2_{D3} = \underbrace{f(r)\left(-dt^2 + \sum_{i=1}^{3} dx_i^2\right)}_{SO(1,3) \ltimes \mathbb{R}^4} + \underbrace{h(r)\left(dr^2 + r^2 d\Omega_5^2\right)}_{SO(6)}$$

There is an event horizon at $r=0$

$f(r) = \frac{1}{h(r)} = H^{-\frac{1}{2}}(r)$, $H(r) = 1 + \frac{R^4}{r^4}$, $R^4 = N \cdot \frac{4}{\pi^2} G_N T_3 = 4\pi N g_s \alpha'$

This spacetime is Assymptotic Flaten. As $r \gg R$ $f \sim 1 + O(\frac{R^4}{r^4})$ this is long-range coulomb potential in $\mathbb{R}^6$. As $r \sim R$, deformation of spacetime metric from D3-branes become significant. D3-branes are located at $r=0$, near this origin:

$$ds^2 \approx \frac{r^2}{R^2}(-dt^2 + \vec{x}^2) + \frac{R^2}{r^2}(dr^2 + r^2 d\Omega_5^2)$$

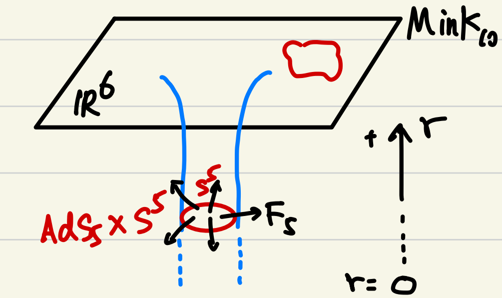
$$= \underbrace{\frac{r^2}{R^2}(-dt^2 + \vec{x}^2) + \frac{R^2}{r^2} dr^2}_{AdS_5} \times \underbrace{+ R^2 d\Omega_5^2}_{S^5}$$

Therefore we have two descriptions of D3-branes

(A) From Dirichlet B.C. of open strings

(B) Spacetime metric ds^2_{D3} and F_5 Flux on S^5. In this description we don't need open string, only closed strings live on.

sourced by D3



These two descriptions should be equivalent. In 1997, Maldacena considered a special limit of this equvalent the low energy limit and it's kown nowadays as the AdS/CFT Duality.

Energy scale

Low Energy Limit: Fix E, take $\alpha' \to 0$ or vise versa

whatever you think the key is $E^2\alpha' \to 0$.

(A) open string $\Rightarrow$ $\mathcal{N}=4$ SYM $U(N)$ gauge Symmetry

\# of branes

$$\boxed{g_{YM}^2 = 4\pi g_s}$$

closed string $\Rightarrow$ graviton, dilaton, ...

At low energy, open strings decouple from closed String even closed string themselves (gravitaty is so weak), So the effective theory is $\mathcal{N}=4$ SYM and free gravitons

(B) In Curved spacetime, "energy" depends on the coordinate time we use. E in (A) defined w.r.t. t i.e. time at $r=\infty$, AFS observer. At r: local proper time $d\tau = H^{-\frac{1}{4}}dt$

$$\boxed{E_\tau = H^{\frac{1}{4}} E_t}$$

For $r \gg R$: $H \sim 1$, again only free graviton remain.

For $r \ll R$: $H \sim \frac{R^4}{r^4}$, $E^2\alpha' \to 0 \Rightarrow E_\tau^2 \frac{r^2}{R^2}\alpha' \to 0$

$$\Rightarrow E_\tau^2 r^2 (4\pi g_s N)^{-\frac{1}{2}} \to 0$$

This means, for any E_τ, the low energy limit means $r \to 0$. Which means for small r (closed to D3 branes) any energy scale (i.e. massive modes) are allowed.

In conclusion: This (B) description at low energy limit = free gravitons at $r=\infty$ + full string theory in $AdS_5 \times S^5$ at $r=0$ (with flux)

These two sectors should be decoupled, but become coupled as E increases

$N \gg 1$ in SUGRA region.

$$\boxed{\mathcal{N}=4 \text{ SYM with } U(N) \quad = \quad \text{IIB string in } AdS_5 \times S^5}$$

open string gauge vector boson spectrum live on the D3-brane volume (and ϕ^a massless scalar live on the transverse space)

so SYM lives on $\mathbb{R}^{3,1}$ which is also the boundary of AdS_5

AdS/CFT duality

AdS_{d+1} spacetime is a solution of Einstein equation with negative cosmology constant, it has constant curvature

$$\boxed{\begin{gathered} ds^2 = \frac{r^2}{R^2}(-dt^2+d\vec{x}^2) + \frac{R^2}{r^2}dr^2 \overset{z:=R^2/r}{=\!=\!=} \frac{R^2}{z^2}(-dt^2+d\vec{x}^2+dz^2) \\ \mathcal{R} = -d(d+1)R^2 , \quad \mathcal{R}_{\mu\nu\rho\lambda} = -R^2(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}) \end{gathered}}$$

Remark: We take low energy limit before to decouple some d.o.f. such as graviton, massive modes. But onece we claim the AdS/CFT conjecture, these modes have been decoupled, so we don't need to keep low energy limit, i.e. $r \to 0$

But any one of these two coordinates only covers a part of AdS spacetime. The global AdS_{d+1} spacetime can be described as a hyperboloid in a flat Lorenz space-time of signature $(2,d)$:

$$ds^2 = -dX_{-1}^2 - dX_0^2 + d\vec{X}^2, \quad X_{-1}^2 + X_0^2 - \vec{X}^2 = R^2$$

(1) Poincaré coordinates

$r = X_{-1} + X_d$, $x^\mu = R\frac{X^\mu}{r}$, corresponds to $r>0$ patch

the metric agrees before.

(2) Global Coordinate

$$X_0 = R\sqrt{1+r^2}\cos\tau, \quad X_{-1} = R\sqrt{1+r^2}\sin\tau \qquad r\in[0,\infty)$$

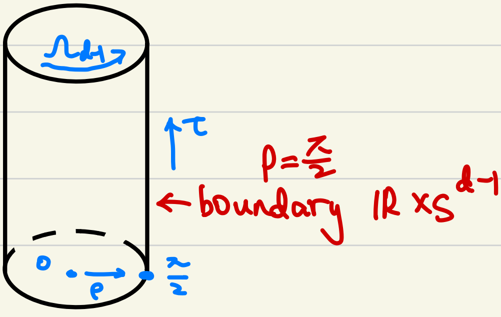
$$X_0^2 + X_{-1}^2 = R^2(1+r^2), \quad \vec{X}^2 = R^2r^2 \qquad \tau\in(-\infty,+\infty)$$

$$ds^2 = R^2\left[-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2\right]$$

Take $r = \tan\rho$, $\rho\in[0,\frac{\pi}{2})$

$$ds^2 = \frac{R^2}{\cos^2\rho}\left(-d\tau^2 + d\rho^2 + \sin^2\rho\, d\Omega_{d-1}^2\right)$$

AdS is conformal equivalent to Cylinder.



Boundary of AdS_{d+1} is $\mathbb{R}\times S^{d-1}$ in global coordinate and $\mathbb{R}^{1,d-1}$ in poincaré coordinate

Symmetries of AdS_{d+1}

isometry : $SO(2,d)$, translation is broken

Actually, this is $D=(d-1)+1$ conformal symmetry. Consider its realization in poincaré coordinate.

- Translation: $x^\mu \longmapsto x^\mu + a^\mu$
- Rotation: $x^\mu \longmapsto \Lambda^\mu{}_\nu x^\nu$
- scaling : $x^\mu \mapsto \lambda x^\mu$, $z \mapsto \lambda z$
- special transformation: $x^\mu \mapsto \dfrac{x^\mu + b^\mu A}{1+2b\cdot x + b^2 A}$

$$z \mapsto \frac{z}{1+2b\cdot x + b^2 A}, \text{ where } b_\mu \text{ is const}, \ A := z^2 + x^2$$

String theory in $AdS_5 \times S^5$.

$AdS_5 \times S^5$ is a homogeneous spacetime which has position independent curvature R. There are only two dimenssion-less parameter in string theory: $\frac{\alpha'}{R^2}$, g_s, or equavalently

$16\pi G_N = (2\pi)^7 g_s^2 \alpha'^4 \Rightarrow g_s \sim \frac{G_N}{R^8}$.

- classical gravity limit: $g_s \to 0$, $\frac{\alpha'}{R^2} \to 0 \Rightarrow$ SUGRA
- classical string limit: $\frac{\alpha'}{R^2}$ fixed, $g_s \to 0$

S^5 is compact, so it is convenient to expand 10 dimensional fields in terms of harmonics on S^5.

$$\Phi(x^\mu, z, \Omega_5) = \sum_l \phi_l(x^\mu, z)\, Y_l(\Omega_5)$$

fields in AdS_5

Gravity is essentially on 5D AdS_5, do dimensional reduction on S^5 the graviton 0-mode on S^5 is Volume of S^5, V_5.

$$\frac{1}{16G_N}\int d^5x\, d^5\Omega \sqrt{-G_{10}}\, R_{10} = \frac{V_5}{16\pi G_N}\int d^5x \sqrt{-G_5}\, R_5$$

$\Rightarrow$ effective 5d Newton Constant:

$$\boxed{G_5 = \frac{G_N}{V_5} = \frac{G_N}{\pi^3 R^5}}$$

$\mathcal{N}=4$ SYM theory

Field content: A_μ, ϕ^i $(i=1,\cdots,6)$, $\underset{\text{Weyl fermions}}{\chi_a^A}$ $(A=1,\cdots 4)$

$$\mathcal{L} = -\frac{1}{g_{YM}^2}\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\phi^i)^2 + [\phi^i,\phi^j]\right) + \text{fermions}$$

$U(N) = SU(N)\times U(1)$, As photon decoupling formulae in QFT, $U(1)$ part always decouples in $U(N)$, so we only need to focus on $SU(N)$ part.

The most important property of $\mathcal{N}=4$ SYM is its β-function vanish, which coms from g_{YM}^2 is dimensionless. This property tells us $\mathcal{N}=4$ SYM is a CFT! The conformal group is exactly $SO(d,2)$, it also have $SO(6)$ symmetry for rotating ϕ^is' indices. Including SUSY, $SO(d,2)\times SO(6)$ bosonic

Symmetries are lifted to super conformal Symmetry

$$\boxed{PSU(2,2|4)}$$

Now we can summarize :

$\mathcal{N}=4$ SYM with $SU(N)$ on $\mathbb{R}^{1,3}$ $=$ IIB String in $AdS_5 \times S^5$ (Poincaré patch)

LHS is 4d CFT, $\mathbb{R}^{1,3} = \partial AdS_5$

RHS is 5d gravity theory

so this relation (conjecture) can be considered as realization of Holographic Principle.

But, one more thing, in quantum gravity space time fluctuates, so what do we really mean by $AdS_5 \times S^5$.

Actually, we should interpret $AdS_5 \times S^5$ as specifying the asymptotic structure of the bulk spacetime.

In AdS/CFT conjecture we always consider semi-classical quantum gravity in the bulk (AdS_5)

Duality Toolbox

IR/UV connection.

Where does the extra dimension come from? As we see before, as we approach the center of D3 brane, namely $r \to 0$, we get the low energy limit of boundary theory. In otherwords, the extra dimension r can be considered as representing the energy scale of the boundary theory

Let us look more closely by using AdS metric:

$$ds^2 = \frac{R^2}{z^2}\left(-dt^2 + dx^2 + dz^2\right)$$

local proper time and proper length:

$$d\tau = \frac{R}{z}\,dt \quad , \quad dl = \frac{R}{z}\,dx$$

This tells you:

$$E \sim \frac{1}{\tau}$$

$$E_{YM} = \frac{R}{z} E_{loc} \quad , \quad d_{YM} = \frac{z}{R} d_{loc}$$

For the bulk process at different z, E_{loc}, d_{loc} remain the same but for viewer on boundary:

$$E_{YM} \propto \frac{1}{z} \quad , \quad d_{YM} \propto z$$

In particular, as $z \to 0$, $E_{YM} \to \infty$, $d_{YM} \to 0$.

$z \to \infty$, $E_{YM} \to 0$, $d_{YM} \to \infty$

AdS

IR $z=0$ ∂AdS

$+\downarrow$

UV $z=\infty$

$\mathcal{N}=4$ YM

UV

$+\uparrow E_{YM}$

IR

IR/UV Connection.

Renormalization Group Flow

"Put IR cutoff in AdS at $z=\varepsilon$" equals to "in the boundary introduce a UV cut off at $\Delta x \sim \varepsilon$, $E_{YM} \sim \frac{1}{\varepsilon}$."

Remark : Holography in global AdS patch.

We described AdS/CFT correspond in Poincaré patch where $\partial AdS_5 = \mathbb{R}^{3,1}$. We can also do this in the global patch, the Only changed thing is $\partial AdS_5 = S^3 \times \mathbb{R}$. This is a non-trivial prediction from holographic principle conjecture, because in D3 brane (open string view) we can only tackle $\mathbb{R}^{3,1}$, but now we argue we can construct the same duality for $S^3 \times \mathbb{R}$ topology. There are some clear differences between these two patchs

Poincare patch	global patch
$r \to 0 \Rightarrow g_{tt} \to 0$	$r \to 0 \Rightarrow g_{tt} \to 1$
$E_{YM} \to 0$	$E_{YM} \to \frac{1}{l_s}$ (radius of S^3)
Spectrum is continue	Has a lowest non-vanish energy
property of CFT on non-compact mfd	property of CFT on compact mfd

Matching of Symmetries.

$\mathcal{N}=4$ SYM

Spacetime: Conformal $SO(4,2)$

Global: $SO(6)$ from $\{\phi^i\}$

SUSY: $\{Q^A_a\}$ 4 Weyl spinor in 4d. And conformal symmetry give you another 4

$\Rightarrow 2^2 \times (4+4) = 32$ real components

IIB string in $AdS_5 \times S^5$

isometry of AdS^5 : $SO(4,2)$

isometry of S^5 : $SO(6)$

one Weyl spinor in 10d

$2^5 = 32$ real components

(but local in SUGRA)

In some extend, isometry of $AdS_5 \times S^5$, $SO(4,2) \times SO(6)$, is a subgroup of Diff^+ group which is a local symmetry so we match global symmetry in CFT and local symmetry in quantum gravity on AdS. But why only isometry, not other subgroup of Diff^+? Recall that in quantum gravity, geometry is fluctuating, so we can only consider $AdS_5 \times S^5$ as an asymptotic background

This subgroup $SO(4,2) \times SO(6)$ is the subgroup which leaves the asymptotic form of the metric invariant, called Large Gauge Transformation (don't become identity at $r=\infty$) this transformation can be consider as "global part" of Diff^+.

The story is more general!

Symmetries in the Field side can be considered as global part (Large gauge symmetry) of the Local Symmetries on the AdS side

CFT in $Mink_d$		AdS_{d+1} quantum Gravity.
Conformal $so(d,2)$	$\Longleftrightarrow$	AdS isometry $SO(d,2)$
$U(1)$ global		$U(1)$ gauge symmetry
global SUSY		local SUSY.

Matching of parameters.

$\mathcal{N}=4$ SYM		IIB on $AdS_5 \times S^5$
g_{YM}^2	$=$	$4\pi g_s$
$\lambda \equiv g_{YM}^2 N$	$=$	$\frac{R^4}{\alpha'^2}$
$SU(N)$	$\sim$	N coincident D3 (flux charges)

Using $16\pi G_N = (2\pi)^7 g_s^2 \alpha'^4$, we obtain:

$$\frac{\pi^4}{2N^2} = \frac{G_N}{R^8} \xrightarrow{\text{Dimension Reduction on } S^5} \frac{1}{G_5} = \frac{V_{S^5}}{G_N}, \; V_{S^5} = \pi^3 R^5 \qquad \boxed{\frac{G_5}{R^3} = \frac{\pi}{2N^2}}$$

Consider (semi-) classical gravity limit. $G_N, \alpha' \to 0$

IIB in $AdS_5 \times S^5$ $\Longrightarrow$ QFT in curve spacetime.

On the SYM side, this means :

$N \to \infty$: Large N Limit

$\lambda \to \infty$ Strong coupling Limit

strong coupling limit is described by classical GR!

YM expansion in $\frac{1}{N^2}$ $\longleftrightarrow$ QG corrections expansion in G_N

YM expansion in $\frac{1}{\sqrt{\lambda}}$ $\longleftrightarrow$ QG corrections expansion in $\frac{\alpha'}{R^2}$

Next, Let's consider classical string limit

$\boxed{\begin{matrix} N \to \infty \\ \lambda \text{ fixed} \end{matrix}}$ $\begin{matrix} \longleftrightarrow & \frac{G_N}{R^8} \to 0 \\ \longleftrightarrow & \alpha' \text{ fixed} \end{matrix}$

t'Hooft Limit. Again, we find Gauge/String duality.

Matching of the spectrum.

Boundary	Bulk
representation of Conformal Group $SO(2,4)$	..... of isometric group
Conformal local operators $\mathcal{O}$	bulk fields
Scalar operators J_α	vector fields A_M
tensor operators $T_{\mu\nu}$	Tensor fields T_{MN}

If there are other symmetries, quantum numbers and representations under them, they should also match.

Actually, we can find this dictionary exactly for $AdS_5 \times S^5$ case

$\mathcal{N}=4$ SYM	IIB in $AdS_5 \times S^5$
$\mathcal{L}_{SYM}$ (regard as an oprator)	dilaton ϕ
$SO(6)$ current J^a_μ	$SO(6)$ gauge field reduced from S^5 A^a_M
enery momentum tensor $T_{\mu\nu}$	Metric pertubations: h_{MN}

Consider deformation of the original boudary CFT

$$\mathcal{L} \to \mathcal{L} + \int d^d x \ \phi_0(x)\, \mathcal{O}(x)$$

$\mathcal{O}(x)$ is a local field in CFT and we call the $\phi_0(x)$ as source of $\mathcal{O}(x)$. The simplest case is $\phi_0(x) \sim$ Const than on the boundary, only coupling of $\mathcal{O}(x)$ is changed.

What does this deformation mean in the bulk?

In a heuristic way, consider

$$\Delta \mathcal{L} = \int d^d x \ \Delta g\, \mathcal{L}_{SYM}$$

$\mathcal{L} \mapsto \mathcal{L} + \delta\mathcal{L}$ equals to $g^2_{YM} \mapsto g'^2_{YM}$, we know that

$$g^2_{YM} \longleftrightarrow g_s = e^{\langle \Phi \rangle} \quad , \langle \Phi \rangle = \Phi_{\partial AdS}$$

where we assume there is no fluctuation of Φ approaching ∂AdS

We also know $\Phi \longleftrightarrow \mathcal{L}_{SYM}$, g^2_{YM} is the source of $\mathcal{L}_{SYM}$ and corresponding to $\langle \Phi \rangle = \Phi_{\partial AdS}$, so in the bulk, what we see is the change of $\Phi|_{\partial AdS}$ as we deforming $\mathcal{L}_{SYM}$

$$\boxed{\begin{aligned}&\int d^dx\, \phi_0(x)\, \mathcal{O}(x) \text{ in boundary theory} \\ &\Longleftrightarrow \text{ bulk field } \phi(x) \text{ dual to } \mathcal{O}(x) \text{ with boundary value } \phi_0(x)\end{aligned}}$$

up to some renormalization factor. This relation is powerful

$$\boxed{\int a_\mu(x)\, J^\mu(x)\, d^dx \longleftrightarrow A_\mu(x,z)|_{z=0} = a_\mu(x)}$$

Since $J^\mu(x)$ is conserved current, so this relation is invariant under $a_\mu \mapsto a_\mu + \partial_\mu \Lambda$ $\Rightarrow$ dynamics of $A_\mu(x,z)$ should also be invariant. This tells you $a_\mu \mapsto a_\mu + \partial_\mu \Lambda$ must be a subset of some gauge symmetries in bulk.

$A_\mu \sim A_\mu + \partial_\mu \tilde{\Lambda}$, A_μ is a gauge field in bulk as we said before

$$\boxed{\int d^dx\, h_{\mu\nu}(x)\, T^{\mu\nu}(x) \longleftrightarrow \text{deform boundary spacetime}}$$

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + dx^2 + dz^2) = \frac{R^2}{z^2} dz^2 + g_{\mu\nu}\, dx^\mu dx^\nu$$

where. $g_{\mu\nu}(z, x^\mu)|_{z\to 0} = \frac{R^2}{z^2}\eta_{\mu\nu}$

So we expect after the deformation : renormalisation factor.

$$g_{\mu\nu}(z, x^\mu)|_{z\to 0} = \frac{R^2}{z^2}(\eta_{\mu\nu} + h_{\mu\nu})$$

If the boundary theory has $T^{\mu\nu} \neq 0$, then the bulk theory must involve gravity. Because $T^{\mu\nu}$ encodes the fluctuation of the bulk spacetime.

Remark: In partition function view $\mathcal{L} \mapsto \mathcal{L} + \int d^dx\, \phi(x)\mathcal{O}(x)$ equals the partition function now depends on $\phi(x)$:

$$Z_{CFT}[\phi(x)] = \left\langle e^{\int d^dx\, \phi(x)\, \mathcal{O}(x)} \right\rangle_E$$

Mass - dimension relation

On gravity side

come from $\frac{1}{2\kappa_0^2}$

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}\left[R - 2\Lambda + 2\mathcal{L}_{matter}\right]$$

where $2\kappa^2 = 16\pi G_{d+1}$, Consider scalar field

$$\mathcal{L}_{matter} = -\frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2 + \text{nonlinear term}$$

Consider small perturbations around AdS, convenient to to Canonically normalize the kinetic term of small perturbations:

AdS metric

$$\Phi \mapsto \kappa\Phi, \quad g_{MN} \mapsto g^{(0)}_{MN} + \kappa h_{MN}$$

Recall that $\kappa^2 \sim G_N \sim \frac{1}{N^2}$, so in the Large N Limit. Φ and $h_{MN} \sim O(1)$, then their pertubations are small.

$\Rightarrow$ nonlinear terms are $O(\kappa)$ or higher, such as cubic term

$$\frac{1}{\kappa^2}\Phi^3 \longrightarrow \frac{1}{\kappa^2}(\kappa\Phi)^3 \sim \kappa\Phi^3 \sim O\left(\frac{1}{N}\right)$$

This means in Leading order, we Only need to consider free theory in AdS.

Remark: why $N \to \infty$? Because we want to use QFT on the AdS side, not type IIB String theory

By our normalization, scalar field's action under small perturbation of AdS is:

$$\boxed{S = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left(g^{MN}\partial_M\Phi\partial_N\Phi + m^2\Phi^2\right) + \mathcal{O}(\kappa)}$$

Where, g^{MN} is the AdS_{d+1} metric. Now you can quantize this theory by techniques of curve spacetime QFT.

E.O.M. : $\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\Phi\right) - m^2\Phi = 0$

$X^M = (x^\mu, z)$, for the translation symmetries in x^μ

$$\Phi(z, x^\mu) = \int\frac{d^dk}{(2\pi)^d}e^{ik\cdot x}\Phi(z,k) \quad , k\cdot x := \eta_{\mu\nu}k^\mu x^\nu$$

$\Rightarrow$ E.O.M. : $\boxed{z^{d+1}\partial_z\left(z^{1-d}\partial_z\Phi\right) - k^2z^2\Phi - m^2R^2\Phi = 0}$

This equation can be solved exactly, and then you can do canonical quantization as we do in flat spacetime QFT

Consider behavior of Φ as $z\to 0$ (near the boundary)

$$\Rightarrow \quad z^2\partial_z^2\Phi + (1-d)z\partial_z\Phi - m^2R^2\Phi = 0$$

By the ansatz $\Phi\sim z^\Delta$:

$$\boxed{\Delta(\Delta-1) + (1-d)\Delta - m^2R^2 = 0}$$

$$\Rightarrow \quad \Delta = \frac{d}{2}\pm\sqrt{\frac{d^2}{4}+m^2R^2} := \frac{d}{2}\pm\nu$$

now we define : $\Delta := \frac{d}{2}+\nu$, $\Delta_- := \frac{d}{2}-\nu = d-\Delta$

As $z\to 0$:

$$\Phi(k,z)\sim A(k)z^{d-\Delta} + B(k)z^\Delta$$

$$\text{FT} \longrightarrow \Phi(x,z)\sim A(x)z^{d-\Delta} + B(x)z^\Delta$$

The exponents are real if $m^2R^2 \geq -\frac{d^2}{4}$. In flat

Called BF bound

Space time we often argue negative m^2 means instability such as tachyon. But in AdS, one can show as far as $m^2R^2 \geq -\frac{1}{4}d$, even though $m^2<0$, the theory is well-define

To do canonical quantization, we need to expand Φ in terms of a complete set of normalizable modes. The normalizability is defined by finiteness of the following klein-Gorden inner product:

$$\boxed{(\Phi_1, \Phi_2) = -i\int_\Sigma dz\, d\vec{x}\, \sqrt{-\gamma}\, n^\mu (\Phi_1^* \partial_\mu \Phi_2 - \Phi_2 \partial_\mu \Phi_1^*)}$$

Where Φ is the Cauchy hypesurface with induced metric γ_{ij} and unit normal vector n^μ. This inner product is independent of choice of Σ if flux leaking through boundary is zero: roughly speaking, independent on time.

$$\boxed{\sqrt{\gamma_z}\, n_z^\mu (\Phi_1^* \partial_\mu \Phi_2 - \Phi_2 \partial_\mu \Phi_1^*) \to 0}$$

as $z \to 0$, $n_z^\mu \partial_\mu = z\partial_\mu$, γ_z is induced boundary metric

Actually, this boundary condition is also important for a well-defined notion of energy in AdS. because it implies the eneary conservation in AdS.

Generally, we can expand a scalar field Φ as:

$$\Phi(x,z) = \sum_n \phi_n(x,z)\, a_n + \phi_n^*(x,z)\, a_n^\dagger$$

But there are many different vacuas in curved spacetime so $(a_n, a_n^\dagger)$ are not uniquely. We can do Bogoliubov transformations to get a new set of annihilation-creation operators that corresponds to different vacua.

For $\nu \in \mathbb{R}$, $\Delta \geq \frac{d}{2}$:

z^Δ mode : always normalizable

$z^{d-\Delta}$ mode : $\begin{cases} \text{normalizable}, & 0 \leq \nu < 1 \\ \text{non-normalizable}, & \nu \geq 1 \end{cases}$

From now on, we will use "normalizable" to a specific quantization, such as when $0<\nu<1$, we can chose $B=0$ to quantize the theory w.r.t. chose $\{z^{d-\Delta}\}$ as the complete set. At this time, we will think z^Δ non-renomalizable, even though it's always normalizable mathematically. We will talk about "standard quantization", $A=0$.

Normalizable modes are used to build up Hilbert space in the bulk, and in AdS/CFT correspondence, they have their boundary counterparts

But non-normalizable modes are not part of Hilbert space. They can only be considered as background. They determine the boundary theory itself. Remember $\Delta \geq \frac{d}{2}$ so z^{Δ} is faster than $z^{d-\Delta}$ to be zero. So the Leading term near the boundary is $A(x) z^{d-\Delta}$ so $A(x)$ is the boundary value of the scalar field, and it will modify the boundary action by $\int d^d x\, A(x)\, \mathcal{O}(x)$:

$$\boxed{\int d^d x\, \phi_0(x)\, \mathcal{O}(x) \Longleftrightarrow \phi_0(x) = \lim_{z \to 0} z^{\Delta-d}\, \Phi(z,x)}$$

Recall that in standard quantization, the non-renormalizable term $A(x) z^{d-\Delta}$ is zero. So if non-renormalizable modes present, they are boundary backgrounds, which means modifying the boundary action by source $A(x)$.

This relation implies that Δ is the scaling dimensions of $\mathcal{O}(x)$. Because under $x^\mu \mapsto \lambda x^\mu$, $z \mapsto \lambda z$, $\mathcal{O}(x)$ and $\phi_0(x)$ will becomes :

$$\phi_0'(x') = \lim_{z' \to 0} z'^{\Delta-d}\, \Phi'(z',x') = \lim_{z \to 0} z^{\Delta-d}\, \Phi(z,x) \cdot \lambda^{\Delta-\lambda}$$
$$= \lambda^{\Delta-d}\, \phi_0(x)$$

$$\int d^d x'\, \phi_0'(x')\, \mathcal{O}'(x') = \int d^d x\, \phi_0(x)\, \mathcal{O}'(x') \cdot \lambda^{\Delta}$$
$$= \int d^d x\, \phi_0(x)\, \mathcal{O}(x) \Rightarrow \boxed{\mathcal{O}'(x') = \lambda^{-\Delta}\, \mathcal{O}(x)}$$

Remark: You can also chose "alternative quantization" chose $B=0$. even "mixed quantization".

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2}$$

① $m=0$, $\Delta = d$, marginal operator

② $m^2<0$, $\Delta<d$, relevant operator

③ $m^2>0$, $\Delta>d$, irrelevant operator

IR $\to$ UV in boundary means $z\to 0$ in bulk.

$$\Phi(z,x) \sim \phi(x) z^{d-\Delta} + \cdots$$

if $\Delta<d$, as $z\to 0$ the leading term is more and more important in bulk theory. so it is relevant operator.

Summerize:

$$\Phi(z,x) \sim A(x) z^{d-\Delta} + B(x) z^{\Delta} + \cdots$$

$$\Delta = \frac{d}{2} + \nu , \quad \nu = \sqrt{\frac{d^2}{4} + m^2R^2}$$

$$\Phi(x,z) \longleftrightarrow \mathcal{O}(x)$$

normalizable modes $\longleftrightarrow$ states

non-normalizable modes $\longleftrightarrow$ action

Such as in standard Quantization:

$$A(x) \longleftrightarrow \int d^dx\, \phi_0(x) \mathcal{O}(x) , \quad A(x) = \phi_0(x)$$

$$B(x) \longleftrightarrow \langle \mathcal{O}(x) \rangle$$

$$m \longleftrightarrow \Delta$$

In alternative quantization:

$$B(x) \longleftrightarrow \int d^dx\, \phi_0(x)\, \mathcal{O}(x) \quad , \phi_0(x) = B(x)$$

$$A(x) \longleftrightarrow \langle \mathcal{O}(x) \rangle$$

$$m \longleftrightarrow d - \Delta$$

This procedure can be also done for vector field and tensor field. The scaling dimension of (massive) vector field is

source term

$$\Delta_v = \frac{d}{2} + \sqrt{\frac{(d-2)^2}{4} + m^2R^2} \ , \ A_\mu \xleftrightarrow{z\to 0} a_\mu + b_\mu z^{d-2}$$

and the scaling dimension of tensor field is

$$\Delta_t = d \ , \quad h_{\mu\nu} \xleftrightarrow{z\to 0} \frac{1}{z^2}(\eta_{\mu\nu} + \delta g_{\mu\nu})$$

source term

Euclidean correlation functions.

Generating Function:

$$Z_{CFT}[\phi(x)] = \langle e^{\int d^dx\, \phi\, \mathcal{O}} \rangle_E$$

If AdS/CFT correspondense is right we expect:

$$\boxed{Z_{CFT}[\phi(x)] = Z_{grav}[\Phi|_{\partial AdS} = \phi]}$$

Because $\mathcal{O}(x) \longleftrightarrow \phi(x)$, $\underline{\phi(x) \Longleftrightarrow \Phi|_{\partial AdS}}$.

this is an unformal, heuristic form, should understood as $\phi(x) = \lim\limits_{z\to 0} \Phi(x,z)\, z^{\Delta - d}$

We don't know how to calculate the RHS, but we know how to do this in semi-classical limit region. ($g_s \to 0$, $\alpha' \to 0$). Recall this limit is strong coupling limit in SYM ($\lambda \to \infty$, $N \to \infty$).

$$Z_{grav}[\Phi|_{\partial AdS} = \underbrace{\phi(x)}_{\text{non-normalizable mode}}] = \int_{\Phi|_{\partial AdS}=\phi(x)} D\Phi\, e^{S_E[\Phi]}$$

$$\underset{\alpha' \to 0}{\overset{g_s \to 0}{\sim}} \exp[S_E(\Phi_c)]$$

Φ_c is the classical solution of E.O.M. with the B.C. $\Phi_c \xrightarrow{z\to 0} z^\delta \phi(x)$ (more precisely form of $\Phi|_{\partial AdS} = \phi(x)$) By AdS/CFT Correspondence we know the leading term of Z_{CFT} as $N \to \infty$, $\lambda \to \infty$ is

$$\boxed{\log Z_{CFT}[\phi] \underset{\lambda \to \infty}{\overset{N \to \infty}{\sim}} S_E[\Phi_c], \quad \Phi_c \longmapsto z^\delta \phi(x)}$$

So we can use AdS/CFT Correspondense to Calculate strongly coupling SYM theory.

$\phi(x)$ is the non-normalizable mode on AdS side, so it should also be cosidered as infinitesimal on CFT side.

$Z_{CFT} = Z_{grav}$, but they are both divergent:

LHS: UV divergence

IR/UV connection

RHS: $z \to 0$, $g_{\mu\nu} \to \infty$, "Volume" divergence near ∂AdS

So we need to renormalize them by add conterterm

$$S_E^{(R)}[\Phi_c] = S_E[\Phi_c]\big|_{z=\epsilon} + S_{ct}[\Phi_c(\epsilon)]$$

must be local functional of Φ_c

After this reglarization we can calculate the n-point correlation functions:

$$\boxed{\begin{aligned}\langle O_1(x_1)\cdots O_n(x_n)\rangle_c &= \frac{\delta^n \log Z_{CFT}^{(R)}}{\delta\phi_1(x_1)\cdots\delta\phi_n(x_n)} \\ &= \frac{\delta^n S_E^{(R)}[\Phi_c]}{\delta\phi_1(x_1)\cdots\delta\phi_n(x_n)}\Big|_{\phi=0}\end{aligned}}$$

Recall Z_{CFT} is the generate function for connected add non-connected Feynman Diagram. But $Z_{CFT} = e^W$, W is the generate function only for connected ones which are our need for on-shell reasons.

Consider 1-point function with source:

$$\langle O(x)\rangle_\phi = \frac{\delta S_E^{(R)}[\Phi_c]}{\delta\phi(x)} = \lim_{z\to 0} z^{d-\Delta} \frac{\delta S_E^{(R)}[\Phi_c]}{\delta\Phi_c(z,x)}$$

$$\overset{\text{H-J eq.}}{\sim} \lim_{z\to 0} z^{d-\Delta}\, \Pi_c(z,x)$$

Π_c is the canonical momentum conjugate to Φ_c treating z as "time"

$$\Phi(x,z) \xrightarrow{z\to 0} A(x) z^{d-\Delta} + B(x) z^{\Delta}, \quad A(x) \leftrightarrow \phi(x)$$

$$\Rightarrow \quad \boxed{\langle O(x) \rangle_\phi = 2\nu B(x)}$$

Now we will prove it for scalar field.

$$S_E = -\frac{1}{2}\int d^{d+1}x\sqrt{g}\,(g^{MN}\partial_M\Phi\,\partial_N\Phi + m^2\Phi^2)$$

$$\Rightarrow \Pi = -\sqrt{g}\, g^{zz}\,\partial_z\Phi$$

$$\Rightarrow \text{E.o.M.} \quad -\partial_M(\sqrt{g}\, g^{MN}\partial_N\Phi) + m^2\Phi = 0$$

$$\Phi(x,z) \sim \Phi(k,z)e^{ik\cdot x} \Rightarrow z^{d+1}\partial_z(z^{1-d}\partial_z\Phi) - k^2z^2\Phi = m^2R^2\Phi$$

Do IBP for S_E:

$$S_E[\Phi_c] = -\frac{1}{2}\int d^{d+1}x\,\Phi_c[m^2\Phi_c - \partial_M(\sqrt{g}\,g^{MN}\partial_N\Phi_c)]$$
$$-\frac{1}{2}\int_0^{+\infty} dz\, d^dx\, \partial_M[\sqrt{g}\,g^{MN}\Phi_c\partial_N\Phi_c]$$

Φ_c is on-shell so the first term vanish:

$$S_E[\Phi_c] = \frac{1}{2}\int d^dx\, \Pi_c\Phi_c\Big|_0^{+\infty} = \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\Phi_c(k,z)\Pi_c(-k,z)\Big|_0^{+\infty}$$

where we used the fact there is only boundary along z.

$z\to 0$: $\Phi_c \to A(x)z^{d-\Delta} + B(x)z^{\Delta} + \cdots$

$$\Pi_c \propto \partial_z\Phi_c \propto -A(d-\Delta)z^{-\Delta} - \Delta B z^{\Delta-1} + \cdots$$

$$\Rightarrow \Phi_c\Pi_c \propto z^{d-2\Delta} + \cdots \to \infty \quad (\text{for } \nu > 0)$$

so $S_E[\Phi_c]$ is divergent. We need to add a counter term as we said before:

$$S_{ct}[\Phi_c(\varepsilon)] = \frac{1}{2}\int\frac{d^dk}{(2\pi)^d} f(k^2)\,\Phi_c(k,z)\,\Phi_c(-k,z)$$

this is the general ansatz we can write satisfying locality on Φ_c. Consider basis of solutions Φ_1, Φ_2

$$\Phi_1 \to z^{d-\Delta} \quad , \quad \Phi_2 \to z^{\Delta}$$

and their corrections in momentum space:

$$\Phi_1(z,k) = z^{d-\Delta}\left(1 + a_1 k^2 z^2 + \mathcal{O}(k^4 z^4)\right)$$

$$\Phi_2(z,k) = z^{\Delta}\left(1 + b_1 k^2 z^2 + \mathcal{O}(k^4 z^4)\right)$$

Only k^{2n} terms for $k \leftrightarrow -k$ symmetry. Similarly

$$\Pi_1 \to -(d-\Delta) z^{-\Delta} \quad , \quad \Pi_2 \to -\Delta z^{\Delta-d}$$

Now expand Π_c and Φ_c by these basis:

$$\Phi_c = A(k)\,\Phi_1 + B(k)\,\Phi_2$$

$$\Pi_c = A(k)\,\Pi_1 + B(k)\,\Pi_2$$

$A^2 := A(k)A(-k)$

$$\Rightarrow S_E[\Phi_c]\Big|_{z=\varepsilon} = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\left[A^2\Phi_1\Pi_1 + B^2\Pi_2\Phi_2 + AB(\Pi_1\Phi_2 + \Phi_1\Pi_2)\right]$$

the first term divergents as $z \to 0$, but we can't counter it directly, because it's not covariant. But we can choose:

only Φ^2 is covariant not Φ_1^2

$$S_{ct} = \frac{1}{2}\int_{z=\varepsilon} \frac{d^dk}{(2\pi)^d} \underbrace{\frac{\Pi_1}{\Phi_1}}_{\sim f(k^2)} \Phi^2$$

$$= \frac{1}{2}\int_{z=\varepsilon} \frac{d^dk}{(2\pi)^d}\left(A^2\Phi_1\Pi_1 + 2AB_1\Pi_1\Phi_2 + B^2\frac{\Pi_1}{\Phi_1}\Phi_2^2\right)$$

Add S_E and S_{ct} together, and we should require the regularity of Φ in the bulk, so Φ must be finite when $z \to \infty$, so is $\partial_z \Phi$. However, we know $\Pi = -\sqrt{g}\, g^{zz} \partial_z \Phi \sim z^{1-d} \partial_z \Phi \to 0$ when $z \to \infty$. Then the term in infinity vanishes, so only $z \to 0$ contributes to the boundary term. Finally, we obtain:

$$\boxed{S_E^{(2)}[\Phi_c] = \frac{1}{2} \int_{z=\varepsilon} \frac{d^d k}{(2\pi)^d}\, 2\nu A(-k) B(k)}$$

Impose the boundary condition: $A(k) = \phi(k)$ we can solve the E.O.M. of Φ_c, for specific k. introduce $r = kz$, the equation equals to:

$$r^{d+1} \frac{d}{dr}\left(r^{1-d} \frac{d}{dr} \Phi(r)\right) - (r^2 + m^2R^2)\Phi = 0$$

so $\Phi(k,z)$ depends on the combination $k \cdot z$. This is Bessel equation, the solution is:

$$\Phi_c \propto z^{d/2} K_\nu(kz)$$

as $z \to 0$

$$K_\nu(kz) = \frac{\Gamma(\nu)}{2}\left(\frac{kz}{2}\right)^{-\nu} + \frac{\Gamma(-\nu)}{2}\left(\frac{kz}{2}\right)^{\nu} + \cdots$$

so $\chi(k) := \dfrac{B}{A} = \dfrac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\dfrac{k}{2}\right)^{2\nu}$ is independent on $\phi(k)$

which implies:

$$S_E^{(R)}[\Phi_c] = \frac{1}{2}\int \frac{d^d k}{(2\pi)^d} 2\nu \chi(k)\phi(k)\phi(-k)$$

$$\Rightarrow \langle \mathcal{O}(k)\rangle_\phi = 2\nu\chi(k)\phi(k) = 2\nu \frac{B}{A}\cdot A = 2\nu B(k)$$

Now we entirely know the assymptotic behavior of Φ_c near the boundary. (in standard quantization)

☆ $$\boxed{\Phi_c(x,z) \sim \underbrace{\phi(x)}_{\text{source}} z^{d-\Delta} + \underbrace{\frac{\langle \mathcal{O}(x)\rangle_\phi}{2\nu}}_{\text{operator}} z^{\Delta} + \cdots}$$

and also the two point function (Green function)

$$\boxed{G_E(k) := \langle \mathcal{O}(k)\mathcal{O}(-k)\rangle_{\phi=0} = 2\nu\chi(k) = \frac{\langle \mathcal{O}(x)\rangle_\phi}{\phi(k)}}$$

this is exactly the linear responce theory tell us

Fourier transform to the coordinate space:

$$\boxed{G_E(x) \propto \frac{1}{|x|^{2\Delta}}}$$

which is exactly 2-point CFT correlation function, Δ is the conformal dimension.

Higher point functions.

Because $S_E^{(R)} \sim \phi^2$, you maybe expect $\langle \mathcal{O}^{n\geq 3}\rangle = 0$.

This is obviously not corret, because we naively neglect $\mathcal{O}(\phi^3)$ in $S_E[\Phi_c]$. For $\langle \mathcal{O}^{n\leq 2}\rangle$ it's rational, but

$\langle \mathcal{O}^{n\geq 3}\rangle \sim \mathcal{O}(\kappa)$ is infinitesimal, so we need add higher order correction in $\mathcal{L}_{matter}$. For example:

$$S = -\int d^{d+1}x \sqrt{g}\left[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{2}m^2\Phi^2 + \frac{\lambda}{3}\Phi^3\right]$$

where $\lambda \sim \kappa \sim \mathcal{O}(\frac{1}{N})$, E.O.M. becomes:

$$\Box\Phi - m^2\Phi - \lambda\Phi^2 = 0$$

with B.C. $\lim_{z\to 0} z^{\Delta-d}\,\Phi(x,z) = \phi(x)$.

One can solve this equation perturbatively in $\phi(x)$

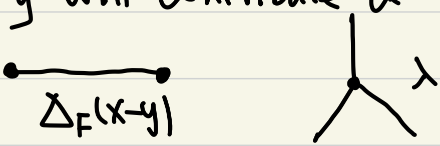
$$\Phi_c = \Phi_1 + \Phi_2 + \cdots$$

$$\Rightarrow S[\Phi_c] = S_2[\phi] + S_3[\phi] + \cdots$$

for 1-pt function $\langle\mathcal{O}\rangle_{\phi=0} = 0$

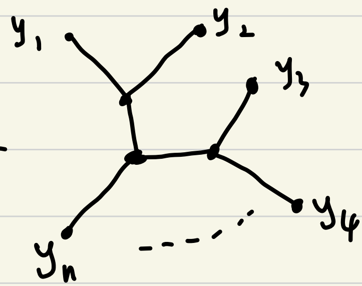
$\Rightarrow \langle\mathcal{O}\rangle_\phi \sim \phi(x) + \cdots$

$S_n[\phi]$ gives you the n-pt functions. But we can use Feynman diagrams to simplify the calculation. Recall in flat spacetime ϕ^3-theory. We treat $\lambda\phi^3$ as perturbative term and contribute a 3-vertex, and the free theory will contribute a free propagator:



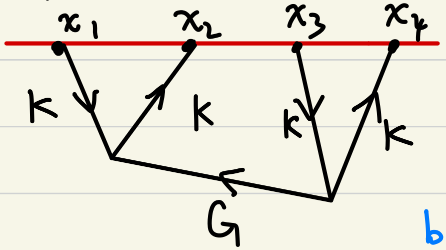
In flat spacetime, because the translation symmetry we always use the Feynman rule in momentum space. Now we need to use the Feynman rule in coordinate

Space

$$\langle \Phi_1(y_1) \cdots \Phi_n(y_n) \rangle =$$ 

$+ \cdots$

Now back to AdS, one major difference is that the sources $\phi(x)$ lie at the boundary:



There are tree different types of propagators in AdS

bulk-to-bulk G

boundary-to-bulk, bulk-to-boundary K

- Bulk-to-bulk propagator.

$$\boxed{(\Box^2 - m^2)\, G(z, x; z', x') = \frac{1}{\sqrt{g}} \delta(z - z') \delta^{(d)}(x - x')}$$

which is the counterpart of standard flat space prop.

And G should also satisfy the B.C.:

$$\boxed{\begin{aligned} G(z, x; z', x') &\sim z^{\Delta} \quad \text{as } z \to 0 \\ G(z, x; z', x') &\sim z'^{\Delta} \quad \text{as } z' \to 0 \\ G(z, x; z', x') &\sim \text{regular} \quad \text{as } z, z' \to \infty \end{aligned}}$$

- Bulk-to-boundary (or bulk-to-boundary) propagator:

$$\boxed{(\Box - m^2)\, K(z,x;x') = 0}$$

and the B.C.

$$\boxed{K(z,x;x') \sim z^{d-\Delta}\delta^{(d)}(x-x') \quad \text{as} \quad z\to 0}$$

because then

$$\Phi(x,z) = \int d^dx'\, K(z,x;x')\,\phi(x') \xrightarrow{z\to 0} z^{d-\Delta}\phi(x)$$

will have right B.C. The Feynman rule is we calculate the bulk correlation function but all the sources on the boundary. And we need to distinguish K and G

$$\boxed{\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_2)\rangle = \langle \Phi_1(x_1)\cdots\Phi(x_n)\rangle}$$

By Feynman diagrams, actually we can capture information beyond semi-classical saddle point by loop diagrams:

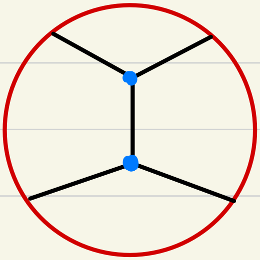
$$Z_{CFT} = \int_{\Phi|_{\partial AdS}=\phi} D\Phi\, e^{S_E[\Phi]} = e^{S_E[\Phi_c]}\int D\delta\, e^{S_E[\Phi_c+\delta]-S_E[\Phi_c]}$$

expand around saddle-point

tree-level

loop-level

You maybe see in other Feynman diagrams like this:



the boundary has S^n topology, not $\mathbb{R}^n$. Because one can show in EAdS $z\to\infty$ corresponds to a single boundary point. $\{z=0\}\cup\{z=\infty\}\simeq S^n$.

So, actually. in EAdS/CFT, CFT on the boundary is defined on S^4 not $\mathbb{R}^4$.

The explicit form of K and G for massive scalar field

$$G(z,x;z',x') = C_\Delta \eta^{-\Delta} F_1\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \frac{\Delta}{2}+1, \frac{1}{\eta^2}\right)$$

$$\eta := \frac{z^2+z'^2+(x-x')^2}{2zz'}, \quad C_\Delta := \frac{2^{-\Delta}\Gamma(\Delta)}{(2\Delta-d)\pi^{d/2}\Gamma(\Delta-d/2)}$$

and you can find expressions of K^Δ in hep-th/9804058

$$K_\Delta(z_0,\vec{x};\vec{x}') = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta-\frac{d}{2})}\left(\frac{z_0}{z_0^2+|\vec{x}-\vec{x}'|^2}\right)$$

there is a relation between K and G:

$$\lim_{z'\to 0} G_B(z,x;z',x') = \frac{z'^\Delta}{2\nu} K(z,x;x')$$

This relation will also cause:

$$\boxed{\langle \mathcal{O}_1(x_1)\cdots \mathcal{O}_n(x_n)\rangle = \lim_{\{z_i\}\to 0} \prod_{i=1}^{n} (2\nu_n z_n^{-\Delta_n}) \times \langle \phi_1(z_1,x_1)\cdots\phi_n(z_n,x_n)\rangle}$$

This formula doesn't surprise us too much, because it's natural to expect:

$$\langle \Phi(x_1)\cdots\Phi(x_n)\rangle \propto \lim_{z_1\to 0}\cdots\lim_{z_n\to 0}\langle \Phi(z_1,x_1)\cdots\Phi(z_n,x_n)\rangle$$

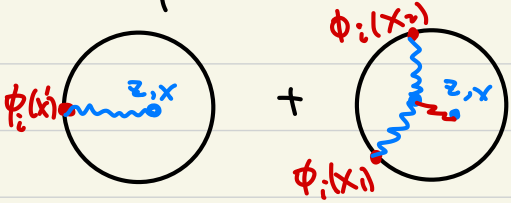
all sources on the boundary   all sources in the bulk.

Let's calculate some explicit examples.

Actually we can consider a more general ϕ^3 theory:

$$S = \frac{1}{2}\int d^{D+1}x\sqrt{-g}\left[\sum_{i=1}^{3}\left((\partial\Phi_i)^2 + m_i^2\Phi_i^2\right) + b\,\Phi_1\Phi_2\Phi_3\right]$$

The Feynman rules in the bulk (i.e. Witten diagrams) come from the perturbative solution of Φ_i:

$\Phi_i(z,x) =$ 

$+ \cdots$

Which means :

integral on the boundary

integral in the bulk

$$\Phi_i(z,x) = \int d^dx' K_{\Delta_i}(z,x;x')\,\phi_i(x') + b\int d^dx'\,dz'\sqrt{-g}\,\times$$
$$\times\, G^{\Delta_i}(z,x;z',x') \times \int d^Dx_1 \int d^Dx_2\, K_{\Delta_j}(z,x;x_1)\,K_{\Delta_k}(z,x;x_2)\,\phi_j(x_1)\,\phi_k(x_2)$$

Φ_i is the bulk field and ϕ_i is its boundary field.

Then you can put this perturbative expantion into $S[\Phi]$, and take fractional derivative, what you get are feynman rules of these witten diagrams.

$$\boxed{\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle_{\phi=0} = b\int d^Dx\,dz\sqrt{-g}\prod_{i=1}^{3} K_{\Delta_i}(z,x;x_i)}$$

Until now, everything goes well, but in fact for K-K modes from S^5 compactification, we have handreds of Feynman rules to use and too many Witten diagrams to add. The entire correlation function is very hard to calculate, even for tree-level. We will only be satisfied with this simple toy model.

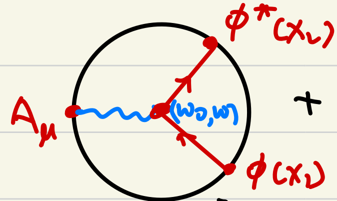
Similarly, you can also construct Feynman rules for gauge fields.

$$S_{bulk} = \int d^{d+1}x \sqrt{-g}\left[-\frac{1}{4}F_{AB}F^{AB} + \frac{1}{2}g^{AB}(\partial_A + iA_A)\phi^*(\partial_B - iA_B)\phi + m^2|\phi|^2\right]$$

(SQED in bulk)

for example:

(*) $\langle \mathcal{O}_\Delta(x_1)\, \mathcal{O}^*_\Delta(x_2)\, J^M(x_3)\rangle =$



$+ \cdots$

$$= -i\int \frac{d^d w\, dw_0}{w_0^{d+1}}\, g^{AB}(w)\left[K_\Delta(w,x_1)\overleftrightarrow{\frac{\partial}{\partial w^B}}K_\Delta(w,x_2)\right]K_A^M(w,x_3)$$

where K_A^M solves the bulk Maxwell equation is the bulk-to-boundary propagator for the gauge boson:

$$K_A^M(w,\vec{x}) = C_D\frac{w_0^{d-1}}{|w-x|^{2(d-1)}}J_A^M(w-x)$$

$$J_A^M(x) := \delta_A^M - 2\frac{x_A x^M}{x^2} = x'^2\frac{\partial x_A}{\partial x'_M}$$

$x_A := \frac{x'^A}{x'^2}$ so J_A^M is the Jacobian for the inversion transformation. More complicate examples can be found in hep-th/9804058. SU(4) gauge anomaly was analyzed in this paper, too.

We use the defination in (*):

$$A \overleftrightarrow{\partial} B := A \overrightarrow{\partial} B - A \overleftarrow{\partial} B$$

Skipping over the technical detals, the answer is:

$$\boxed{\begin{aligned} &\langle O_\Delta(x_1) O_\Delta^*(x_2) J^\mu(x_3) \rangle \\ &\quad = S^\mu(x_1, x_2, x_3)\, \eta \frac{(\Delta - \frac{d}{2})\Gamma(\frac{d}{2})\Gamma(d)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})} \\ &S^\mu(x_1, x_2, x_3) := \frac{1}{x_{12}^{2\Delta - d + 2}} \left(\frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2} \right) \frac{1}{x_{13}^{d-2} x_{23}^{d-2}} \end{aligned}} \quad (**)$$

Under $U(1)$ transformation:

$$\delta_\Lambda O = i\Lambda O, \quad \delta_\Lambda O^* = -i\Lambda O, \quad \delta_\Lambda S = \int \partial_\mu J^\mu \Lambda$$

Then we have Ward-Takahashi equation

$$\begin{aligned} 0 = \delta_\Lambda \langle O O^* \rangle &= \delta_\Lambda \left[\int D\phi\, e^{-S} O_\Delta(x_1) O_\Delta^*(x_2) \right] \\ &= -\left\langle \left(\int d^D x_3\, \partial_\mu J^\mu(x_3) \Lambda(x_3) \right) O_\Delta(x_1) O_\Delta^*(x_2) \right\rangle \\ &\quad + \langle i\Lambda(x_1) O(x_1) O^*(x_2) \rangle + \langle O_\Delta(x_1) (-i\Lambda(x_2) O_\Delta^*(x_2)) \rangle \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial x_3^\mu} \langle O_\Delta(x_1) O_\Delta^*(x_2) J^\mu(x_3) \rangle = i(\delta(x_{13}) - \delta(x_{23})) \langle O_\Delta(x_1) O_\Delta^*(x_2) \rangle$$

Using (**), we get

$$\boxed{\langle O(x_1) O^*(x_2) \rangle = \eta \frac{2\Delta - d}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta - \frac{d}{2})} \frac{1}{x_{12}^{2\Delta}}}$$

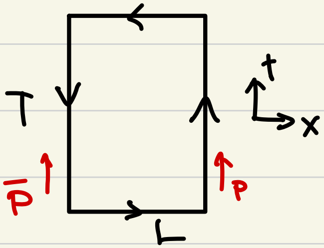
We can use conformal symmetry to obtain $\langle O O^* \rangle \sim \frac{1}{x_{12}^{2\Delta}}$. By AdS/CFT, we calculate the coefficient exactly.

Wilson Loop = ∂(Minimal surface)

Wilson Loops: (in Poincare patch of AdS, not EAdS)

$$W_R[C] = \mathrm{Tr}\, P \exp\left[i\oint_C A_\mu dx^\mu\right], \quad A_\mu = A_\mu^a T_R^a .$$

The physical meaning is phase factor associated with transparting an "external" partical in a R-rep along C (A-B effect) We will work in fundamental Rep.



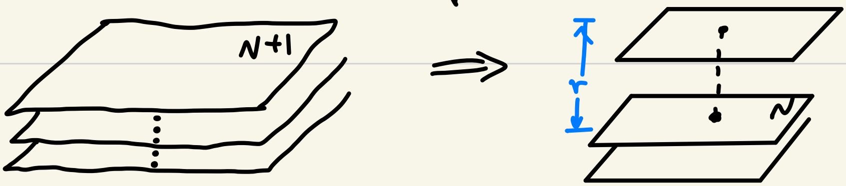
$\underline{T \gg L}$

$$\langle W(C)\rangle \sim e^{-iET} \quad (E = 2M + V(L))$$

V(L) can be interpreted as potential between an external particle and anti-particle

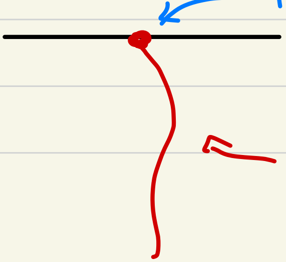
Now we will show how to calculate $\langle W(C)\rangle$ in $\mathcal{N}=4$ SYM using AdS/CFT correspondens. But all things in $\mathcal{N}=4$ SYM are living in the adj. rep. so how to introduce fundamental rep.? And we also need the gravity description of such an external particle.

For the first question, suppose we have $N+1$ stacking D3 branes. If we separate one of them along one perpendicular direction for distance $|\vec{r}|$:



The gauge symmetry will break from $SU(N+1)$ to $SU(N)\times U(1)$
Consider a string with two end points located on the separated D3 brane and the rest N ones respectively. This string corresponds to a fluctuation field living on those D3 branes in $SU(N)$ fundamental representation. It's mass is $M = \frac{|\vec{r}|}{2\pi\alpha'} \neq 0$ for symmetry breaking.

Now consider low energy limit $\alpha' \to 0$, $R \to 0$ but keeping $\frac{R}{\alpha'}$ finite. In the resulted gravity side, one finds only one D3 brane in $AdS_5 \times S^5$ which located at $\vec{r}$ and the other N D3 brans disappeared at $r=0$. Take $r \to \infty$ (i.e. to the boundary) limit. In this case the external particle in fundamental representations with $M \to \infty$. This particle in $\mathcal{N}=4$ SYM can be intepreted in bulk as a "string" hanging from the AdS boundary to deep interior and the hanging point is the location of the particle.



So praralle transport of such an particle in SYM is something like pulling open string in AdS. But SYM also has scalar fields so the wilson loop should be corrected:

$$\boxed{W(c) = \mathrm{Tr}\, P\exp\left[i\oint_C ds\left(A_\mu \frac{dx^\mu}{ds} + \vec{n}\cdot\vec{\Phi}\sqrt{\dot{x}^2}\right)\right]}$$

$\vec{n}$ is a unit vector on S^5. This formale indicates the string ending on a D3 brane with collective coordinate $\vec{\Phi}$, so it must be coupled with scalar fields. In Ads picture the wilson loop on the boundary C should be the boundary of string's worldsheet: $C = \partial\Sigma$

We will expect this from Ads/CFT conjecture:

$$\boxed{\langle W(c)\rangle = Z_{string}[\partial\Sigma = c]}$$

Consider $g_s \to 0$, $\alpha' \to 0$ limit. Then RHS is tractable by saddle-point approximate.

$$\langle W(c)\rangle \sim \exp[i S_{cl}(\partial\Sigma = c)]$$

Examples:

1. a static particle.

On the gauge side : $W = e^{-iMT}$

on the gravity side : $X^i(t,r) = \text{const}$, $\psi = 0$

Remark : In any condition, the classical solution of fermion is always zero.

Using reparametrization freedom on the world sheet, one can choose the coordinate on world sheet $\sigma^\alpha = (\tau, r)$

then $\partial_t X^t = 1$, $\partial_r X^r = 1$, $\partial_\alpha X^i = 0$

$$ds^2_{ws} = h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta \overset{X^i=0}{=\!=\!=} -\frac{r^2}{R^2} d\tau^2 + \frac{R^2}{r^2} d\sigma^2$$

$$\Rightarrow S^{cl}_{NG} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-h} = -\frac{1}{2\pi\alpha'} \int_0^T dt \int_0^{+\infty} dr = -\frac{1}{2\pi\alpha'} T\Lambda$$

Where Λ is the cutoff of r, recall $M = \frac{|\vec{r}|}{2\pi\alpha'} = \frac{\Lambda}{2\pi\alpha'}$

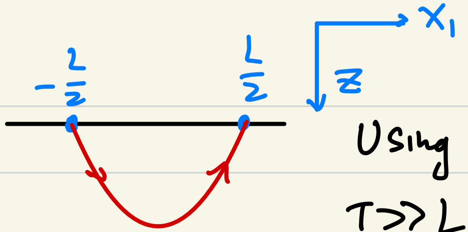
which agrees AdS/CFT Correspondense. In terms z

$z = \frac{R^2}{r} \Rightarrow M = \frac{1}{2\pi\alpha'} \frac{R^2}{\varepsilon} = \frac{\sqrt{\lambda}}{2\pi\varepsilon}$, λ is t'Hooft coupling.

This corresponds to self energy in strong coupling of CFT on the boundary.

2. rectangular loop $T \gg L$

The beginning of this section we calculate this in gauge side perturbatively. But as $\lambda \to \infty$ we don't know how to calculate. We can calculate it by AdS/CFT.



Using reparametrization, choose $\tau = t, \sigma = X^1$

$T \gg L$, translation sym. in time requires

$X^{i\neq 1}(\tau,\sigma) = \text{Const}$, $Z(\tau,\sigma) = z(\sigma)$, $z(\pm\frac{L}{2}) = 0$

$$ds_{ws}^2 = \frac{R^2}{z^2}\left(-d\tau^2 + (1+z'^2)d\tau^2\right)$$

$$\Rightarrow \quad S_{NG} = -\frac{R^2}{2\pi\alpha'} T \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{d\sigma}{z^2}\sqrt{1+z'^2} = -\frac{R^2}{\pi\alpha'} T \int_0^{\frac{L}{2}} \frac{d\sigma}{z^2}\sqrt{1+z'^2}$$

Now we need to extremize it to find $z(\sigma)$ with b.c. $z(\pm\frac{L}{2}) = 0$. E-L equation:

$$\frac{d}{d\sigma}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} = 0 \Rightarrow \frac{2+2z'^2+zz''}{z^3(1+z'^2)^{3/2}} = 0$$

$$\Leftrightarrow \quad -\frac{1}{z'}\frac{d}{d\sigma}\frac{1}{z^2\sqrt{1+z'^2}} = 0 \Rightarrow \frac{1}{z^2\sqrt{1+z'^2}} = \text{const}$$

We expect parity symmetry $z(\sigma) = z(-\sigma) \Rightarrow z(0) = z_0$

$z'(0) = 0 \Rightarrow z'^2 = \frac{1}{z^4 z_0^4} - 1$, it's also very hard to solve, but we can obtain $z(\sigma)$'s inversed function:

$$\sigma = \pm\int_{z_0}^{z} z_0 \, d\tilde{z} \frac{\tilde{z}^2}{\sqrt{1-z_0^2\tilde{z}^4}}$$

The solution for $\sigma > 0$ can be written as:

$$\sigma = z_0 \int_{z/z_0}^{1} dy \frac{y^2}{\sqrt{1-y^4}}$$

Using the fact $z(\frac{L}{2}) = 0$, we can solve z_0

$$\frac{L}{2} = z_0 \int_{\frac{L}{2z_0}}^{1} dy \frac{y^2}{\sqrt{1-y^4}} \overset{L\to 0}{\approx} z_0 \frac{B(\frac{1}{2},\frac{3}{4})}{4}$$

$$\Rightarrow \quad \boxed{z_0 = L\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}}$$

For Calculate S_{NG}, we don't need the explicit solution of $z(\sigma)$:

$$S_{NG} = -\frac{R^2}{2\pi\alpha'} T \int_0^{\frac{L}{2}} \frac{d\sigma}{z^2}\sqrt{1+z'^2} = -\frac{\sqrt{\lambda}}{\pi} T \int_0^{\frac{L}{2}} \frac{d\sigma}{z^2}\sqrt{1+z'^2}$$

$$= -\frac{\sqrt{\lambda}}{\pi} T \int_0^{z_0} \frac{z_0^2}{z^2\sqrt{z_0^4-z^4}} dz = -\frac{\sqrt{\lambda}}{\pi z_0}\int_0^1 \frac{dy}{y^2\sqrt{1-y^4}}$$

This integral divergent, but we can introduce cutoff:

$$S_{NG}/T = -\frac{\sqrt{\lambda}}{\pi z_0}\int_{\varepsilon/z_0}^1 \frac{dy}{y^2\sqrt{1-y^4}}$$

$e^{iS_{NG}} \sim e^{-iET} \sim e^{-i(V+2M)T}$ the M of particle and anti particle are both $\frac{\sqrt{\lambda}}{2\pi\varepsilon}$

$$\Rightarrow V(L) = \frac{\sqrt{\lambda}}{\pi z_0}\int_{\varepsilon/z_0}^1 \frac{dy}{y^2\sqrt{1-y^4}} - \frac{\sqrt{\lambda}}{\pi\varepsilon}$$

$$= \frac{\sqrt{\lambda}}{\pi z_0}\left(\int_{\varepsilon/z_0}^1 \frac{dy}{y^2\sqrt{1-y^4}} - \int_{\varepsilon/z_0}^1 \frac{dy}{y^2} - 1\right)$$

$$= \frac{\sqrt{\lambda}}{\pi z_0}\left(K(i) - E(i)\right)$$

K, E is the complete elliptic integral of the first and second kind respectively. $K(i) = \frac{\sqrt{2\pi}}{8\pi}\Gamma^2(\frac{1}{4}) = E(i) - \frac{\pi\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})}$

$$\boxed{V(L) = -\frac{\sqrt{\lambda}}{L}\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}}$$

As we have seen, the calculation of wilson loop is translated to a minimal (world sheet) surface problem in bulk view.

Finite T on $\mathbb{R}^d$ = Black Brane

Finite temperature generalization

Thermal gas in AdS

It can be described by Euclidean AdS metric

$$ds^2 = \frac{R^2}{z^2}(d\tau^2 + dz^2 + d\vec{x}^2), \quad \tau \sim \tau + \beta$$

Bosons are periodic in τ, fermions are anti-periodic. But this metric has singularity at $z = \infty$, and there is instability comes from stringy calculation. So thermal gas in AdS can not correspond to termal states in SYM. Black Hole maybe a reasonable choice. But now we work in AdS not EAdS, the topology of boundary is $\mathbb{R}^d$, BH have Horizon withe S^d topology, so we need to consider

Black p-brane:

$$\boxed{ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + d\vec{x}^2 + \frac{1}{f(z)}dz^2\right), \quad f(z) = 1 - \frac{z^d}{z_0^d}}$$

With temperature and entropy

$$T = \frac{d}{4\pi z_0}, \quad S_{BH} = \frac{A_3}{4G_5}, \quad A_3 = \frac{R^3}{z_0^3}\int dx_1 dx_2 dx_3$$

define the entropy density as:

$$s = S / \int dx_1 dx_2 dx_3 = \frac{R^3}{z_0^3}\frac{1}{4G_5} = \frac{\pi^2}{2}N^2 T^2 \qquad \left(\frac{G_5}{R^3} = \frac{\pi}{2N^2}\right)$$

This is the temperature on the boundary, so this entropy corresponds to entropy of SYM as $\lambda \to \infty$

As $\lambda \to 0$, the calculation of entropy parallels to calculation for free bosonic / fermionic massless gas. each bosonic d.o.f contributes $\frac{2\pi^2}{45} T^3$, and $S_F = \frac{7}{8} S_B$. for fermionic d.o.f (You need to consult statical physics book about Black Body Radiation) In SYM there are one A_μ with two possible helicity and six ϕ^i with spin-0. They also have N^2-1 color d.o.f. So there are totally $8(N^2-1)$ bosonic d.o.f in SYM. Fermionic d.o.f is the same as bosonic d.o.f for SUSY.

$$S_{\lambda=0} = \left(8 + 8 \times \frac{7}{8}\right) \times \frac{2\pi^2}{45} T^3 (N^2-1) \overset{N\to\infty}{\sim} \frac{2}{3} \pi^2 N^2 T^3$$

$$\Longrightarrow \quad \boxed{\frac{S_{\lambda=\infty}}{S_{\lambda=0}} = \frac{3}{4}}$$

many examples of CFT duals are known in $d=4$ which have

$$\frac{S_{strong}}{S_{weak}} = \frac{3}{4} h \quad , \quad \frac{8}{9} \leq h \leq 1.09$$

This agreement can be also considered as a check of AdS/CFT conjecture.

Finite T on S^d = BH + TAdS

Now, consider CFT on the sphere. (such as EAdS/CFT)

Finite temperature CFT on the sphere is very different from the case on $\mathbb{R}^d$. For CFT on $\mathbb{R}^d$, T is the only scale, scaling invariance provides this theory independents on temperature, But for CFT on S^d radius of the sphere is also a scale, which implies this theory depends on RT.

For theory on the sphere, termal states should correspond to Black Hole with metric:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_{d-1}^2$$

Where

$$f(r) = 1 + \frac{r^2}{R^2} - \frac{\mu}{r^{d-2}}$$

, μ is a const. related to BH mass

notice this solution discribe BH in AdS, not Mink.

As $R \to \infty$, the solution degonerate to d-dimentional Schwarzschild BH. The horizon is located at $r = r_0$, where $f(r_0) = 0$. The temperature is given by:

$$\boxed{\beta = \frac{4\pi}{f'(r_0)} = \frac{4\pi r_0 R}{d r_0^2 + (d-2)R^2}}$$

Further more, thermal states on the sphere can also correspond to thermal gas in AdS. Because

For S^d boundary topology, we need to describe bulk AdS by global coordinates:

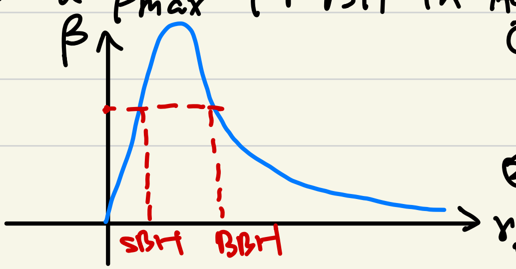
$$ds^2 = -\left(1+\frac{r^2}{R^2}\right)dt^2 + \frac{dr^2}{1+r^2/R^2} + r^2 d\Omega_{d-1}^2$$

Then we can wick rotate t to $-i\tau$ to obtain thermal gas in AdS. This will introduce singularity in poincaré patch, but we can safely require a periodicity, $\tau \sim \tau + \beta$ in global patch. After doing that, the local proper size of τ-circle is

$$\sqrt{1+r^2/R^2}\,\beta \geq \beta$$

Unlike in poincaré patch, where the local proper size of τ-circle goes to 0 when $z \to \infty$. Here, as long as the temperature is not very hight (i.e. β isn't very small), thermal gas is perfectly defined. So thermal state on S^d can be mapped to thermal gas. (at least at low temperature)

Let's talk more about the BH (in AdS). Here is a β_{max} for BH in AdS.



① Only when $T \geq T_{min}$, there exist a BH solution

② $T \neq T_{min}$, It gives you two different BH solutions

As we can see, there are three possible thermal objects in AdS, thermal gas, SBH and BBH. When $T < T_{min}$, only thermal gas survive It implies thermal states on the boundary have 3 phases, we should find the one with lowest free energy

$$e^{-\beta F} = Z_{CFT} = Z_{grav} = \int D\Phi\, e^{S_E[\Phi]} \sim e^{S_E[\Phi_c]}$$

$$\Rightarrow \qquad \boxed{F = -\frac{1}{\beta} S_E[\Phi_c]}$$

so, we should find the solution with largest S_E. This is a conclusion from the boundary side and it can also be analysed on gravity side

$$Z_{grav} = e^{S_E}\big|_{TAdS} + e^{S_E}\big|_{SBH} + e^{S_E}\big|_{BBH}$$

where clearly the solution with largest S_E dominates

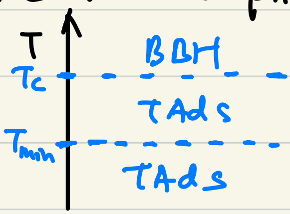
One can show that:

$$S_E(TAdS) = 0 \cdot N^2 + O(N^0)$$

$$S_E(BBH) > S_E(SBH) \sim O(N^2)$$

so SBH will never dominate. In fact, there exist a critical temperature T_c. when $T_{min} < T < T_c$, $S_E(BBH) < 0$, so TAdS dominates in this region; one the other hand, in $T > T_c$ region, $S_E(BBH) > 0$, BBH dominates

Here is the phase diagrame



There is a phase transition at $T = T_c$, and it's a first order phase transition. Because the free energy jumps from $O(N^0)$ to $O(N^2)$, which means the derivative of F is not continuous. This phase transition is called **Hawking - Page transition**.

$\mathbb{R}^{d-1}$ is the R goes to infinity limit of S^{d-1}. We said finite temperature CFT on sphere is parametrized by $R \cdot T$:

$R \to \infty$, T fixed $\xleftrightarrow{RT = \text{Const}}$ R fixed, $T \to \infty$

$\Downarrow$ $\Downarrow$

finite temperature CFT on $\mathbb{R}^{d-1}$ $\qquad$ high temperature CFT on S^{d-1}

$\nwarrow\!\!\searrow$ BH $\swarrow\!\!\nearrow$

So this tells why termal states on $\mathbb{R}^{d-1}$ only corresponds to BH, more precisely Black Brane.

Finite μ = charged BH

$\mathcal{N}=4$ SYM has SO(6) global symmetry, which means it has many charges. For example, we can choose one of the U(1) subgroup and ture a chemical potential for that U(1). The grand canonical partition function is defined as

$$\Xi = \mathrm{Tr}\left(e^{-\beta H - \beta\mu Q}\right)$$

Where Q is the conserved charge for U(1). In field theory on the boundary, this corresponds to deforming the action by

$$\delta S = \int d^4x \, \mu J^0$$

On gravity side, we should turn on the non-normalizable modes for the gauge field A_μ dual to J^μ with the boundary condition below

$$\lim_{z\to 0} A_0(z,x) = \mu \,, \quad \lim_{z\to 0} A_i(z,x) = 0$$

The bulk geometry with finite chemical potential can then be found by solving Einstein-Maxwell eq. with the boundary condition above. Naturally, you will expect the solution is charged BH in AdS, e.g. d=4

$$\boxed{\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \\ f &= 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} + \frac{r^2}{R^2} \end{aligned}}$$

Holographic Entanglemet Entropy

Entanglement entropy

consider a system Consisted of two sub system A and B with Hilbert space:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B .$$

In state $|\Psi\rangle$, A-B are entangle iff $|\Psi\rangle$ can not be written as a simple product. Entenglement entropy provides a measure to quantity entanglement between A and B. The defination is:

$$\boxed{S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad , \quad \rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|}$$

$S_A = 0$ iff ρ_A describes a pure state i.e $|\Psi\rangle$ can be written as a simple product, there is no entangle-ment between A and B

$$\boxed{\text{For any pure state } |\Psi\rangle,\ S_A = S_B .}$$

Suppose A, B, C are three parts of the system without any intersection between any two of them, we hav.

Subadditivity:

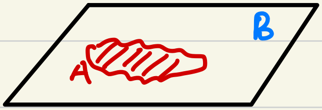
$$|S(A) - S(B)| \leq S(A\cup B) \leq S(A) + S(B)$$

Strong subadditivity:

$$S(A\cup C) + S(B\cup C) \geq S(A\cup B\cup C) + S(C)$$

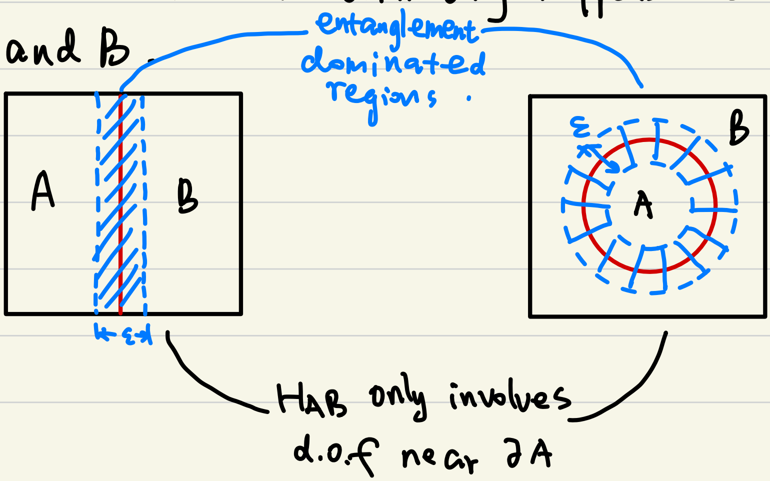
$$S(A\cup C) + S(B\cup C) \geq S(A) + S(B)$$

Many body or field systems can also be formalised by Hamiltonion, they also have quantum. So we can talk about entanglement entropy in these systems too.



$$|\Psi\rangle_{A\cup B} = \sum |\Psi\rangle_A \otimes |\Psi\rangle_B$$

We are interested in ground states of $A \cup B$.
If there is no interaction between A and B, i.e the totall Hamiltonion is $H_{A\cup B} = H_A + H_B$. Then the ground state in unentanglement at any time in evolution.
If we add interactions betwen A and B $H_{A\cup B} = H_A + H_B + H_{AB}$, then the ground state is generally entangled
But in all realistic QFT, H_{AB} is local, which implies that the interaction only happens near the interface of A and B.



One finds, in general, the entanglement entropy of ground states have area law:

$$\boxed{S_A = \gamma \frac{\text{Are}(\partial A)}{\varepsilon^{D-2}} + \text{subleading terms}}$$

Where γ is a constant coefficient, ε is the lattice spacing or short-distance cutoff of QFT. Which characterizes the geometric width of interface. This formula shows entanglement between A and B are dominated by short-range entanglement near ∂A

Subleading terms coming from long-range entangle-ment can provide important chacterization of systems.

① Topological Order in 2+1 dimention.

This system contains long-range correlations not accressible via standard observables such as correlation functions of local operators. But it can be captured by entanglement entropy:

$$S_A = \gamma \frac{L(\partial A)}{\varepsilon} - \delta$$

Subleading terms are characterized by a single constant δ, and it doesn't depend on shape or size of A.

② For 1+1 dimensional CFT, the Area of ∂A is zero. There is no leading contribution, and one finds

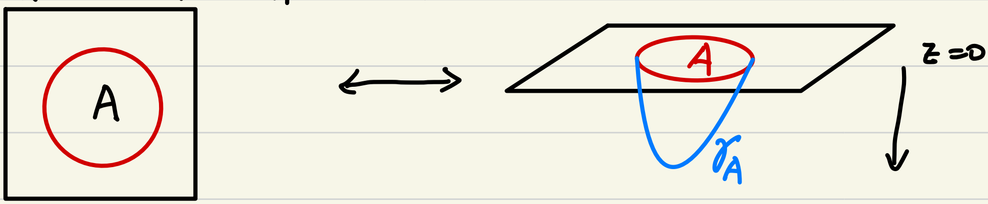
$$\boxed{S_A = \frac{c}{3}\log\frac{L(A)}{\varepsilon}}$$

where c is the central charge.

Remark: The definations of cutoff ε and constant γ depends on the system. And entanglement entropy is calculated in a time slice.

Holographic Entropy.

The question is how to calculate entanglement entropy of CFT_d on AdS_{d+1} side.



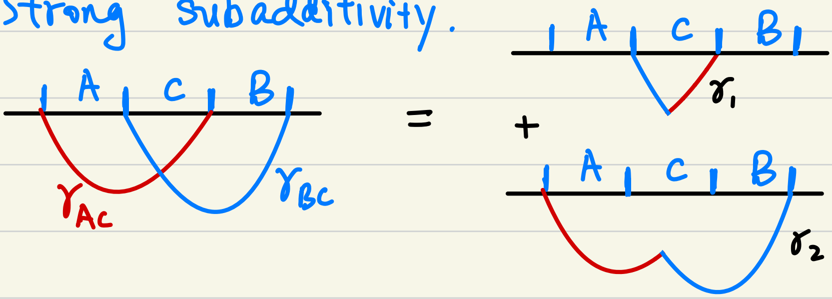
Ryu and Takayanagi guessed the entropy is given by

$$\boxed{S_A = \frac{\text{Area}(\gamma_A)}{4G_N}}$$

Where γ_A is the minimal surface with boundary A.

Some supports for this formula:

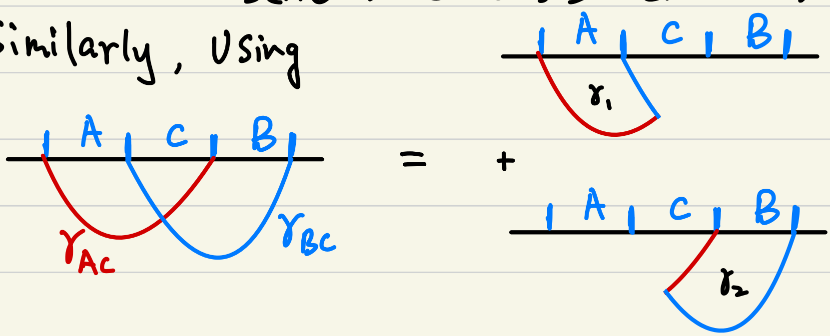
① Strong subadditivity.



$\gamma_{AC}+\gamma_{BC}=\gamma_1+\gamma_2$. By defination of minimal surface. $\gamma_1 \geqslant \gamma_C$, $\gamma_2 \geqslant \gamma_{ABC}$, which implies

$$S(A\cup C)+S(B\cup C)\geqslant S(A\cup B\cup C)+S(C)$$

Similarly, Using



$\gamma_{AC}+\gamma_{BC}=\gamma_1+\gamma_2$, $\gamma_1 \geqslant \gamma_A$, $\gamma_2 \geqslant \gamma_B$. Which implies

$$S(A\cup C)+S(B\cup C)\geqslant S(A)+S(B)$$

This proof is very elegent contrast to highly non-trival proof by von Neumann's formula.

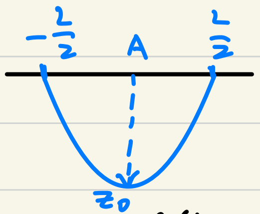
② reproduce known result.

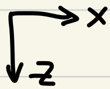
Apply this formula to 1+1 dimensional CFT. For holographic CFT, the central charge is given by

$$\boxed{c = \frac{3R}{2G_N}}$$

on the gravity side, consider constant time slice:

$$ds^2\Big|_{t=0} = \frac{R^2}{z^2}(dx^2+dz^2)$$





$$dl^2 = \frac{R^2}{z^2}(1+x'^2)dz^2$$

B.C. $x(0) = \pm\frac{L}{2}$

$$S(A) = \frac{1}{4G_N} \times 2\int_{\varepsilon}^{z_0} dz \frac{R}{z}\sqrt{1+x'^2}$$

, Extreming it leads to the answer $x = \pm\sqrt{\frac{L^2}{4}-z}$, a semi-circle. $z_0 = \frac{L}{2}$

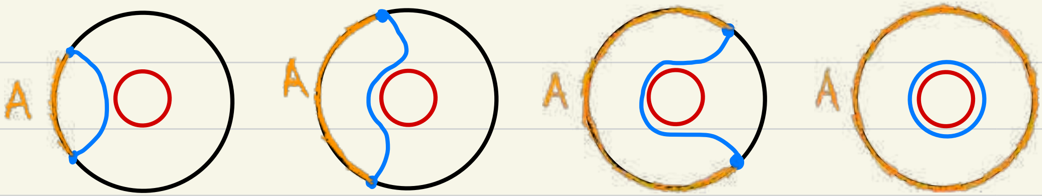
$$\Rightarrow \quad S(A) = \frac{1}{4G_N} \times 2R \times \frac{L}{2}\int_{\varepsilon}^{\frac{L}{2}} \frac{dz}{z}\frac{1}{\sqrt{L^2/4-z^2}}$$

$$\overset{\varepsilon\to 0}{\simeq} \frac{2R}{4G_N}\log\frac{L}{\varepsilon} = \frac{c}{3}\log\frac{L}{\varepsilon}$$

It is exactly what we concluded for 1+1 CFT before.

① relations to BH entropy.

Consider finite temperature CFT (i.e. on S^1) which dual to BH in AdS. If we take A to be the whole boundary space, the minimal surface is just BH horizon, and BH entropy is recovered.



Graphically, as you make A larger and larger, Its minimal surface more and more tends to BH horizon

— The End —