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統計力学 by 川畑幸平

授業ノート

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§1. Landau-Ginzburg Theory.

Ising Model

$$H(\{s_i\}) \equiv -J \sum_{\langle ij \rangle} s_i s_j \quad , \quad s_i = \pm 1$$

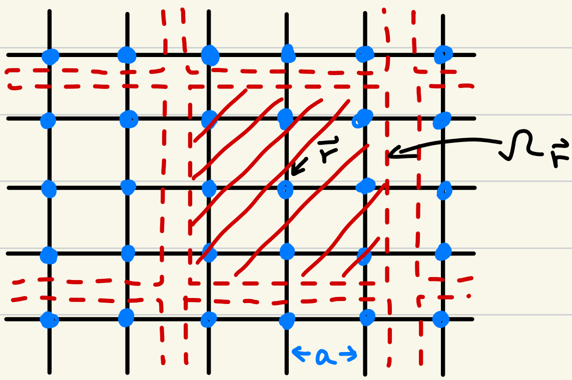
$$Z \equiv \mathrm{Tr}\, e^{-\beta H(\{s_i\})}$$

To capture universality around T_c, a continuous field theory description is useful

$$Z = \int \mathcal{D}\phi(\vec{r})\, e^{-H[\phi(\vec{r})]}$$

real scalar field

Coarse graining



$$\phi(\vec{r}) = \frac{1}{\#(\Omega_{\vec{r}})} \sum_{i \in \Omega_{\vec{r}}} s_i$$

We should choose $\Omega_{\vec{r}}$ such that

- $\Omega_{\vec{r}}$ is sufficiently smaller than the entire sys.
- $\Omega_{\vec{r}}$ should be large enough ($\gg a$)

field theory formulation is meaningless on very short distance scales (we implicitly expect the theory is independent on the coarse graining details)

The effective Hamiltonion $H(\phi(\vec{r}))$ can be derived from lattice model for this simple model, but in general it's impossible. Actually, the Hamiltonion is highly constraint by symmetries and other physics reason.

- Locality : $H[\phi(\vec{r})] = \int d\vec{r}' \mathcal{H}[\phi(\vec{r}')] \supset \phi, \nabla\phi, \cdots$
- $\mathbb{Z}_2$ spin flip symmetry : $H(\{S_i\}) = H(-\{S_i\})$

  $\Rightarrow \mathcal{H}[\phi(\vec{r})] = \mathcal{H}[-\phi(\vec{r})]$
- Around T_c, $\phi(\vec{r})$ should be small (as an order parameter) $\Rightarrow \mathcal{H}[\phi(\vec{r})]$ can be Taylor expanded to $\phi, \nabla\phi, \cdots$

$$\Rightarrow \boxed{\mathcal{H}[\phi(\vec{r})] = \int d\vec{r}\left[t\phi(\vec{r})^2 + u\phi(\vec{r})^4 + \frac{\alpha}{2}(\vec{\nabla}\phi(\vec{r}))^2 + \cdots\right]}$$

Landau-Ginzburg theory

for Ising model, irrelevant

So we don't need to go into the microscopic detail of the theory to derive the $H[\phi(\vec{r})]$, we can constraint it just by some simple and natural argument. However, t, u and α is related to the microscopic detail, for deriving universal behavior such as critical exponent, the exact numeric values of them are irrelevant.

Now suppose ϕ is uniform (independ on $\vec{r}$)

$$Z = \int d\phi\, e^{-V\cdot\mathcal{H}(\phi)} \xrightarrow[\text{saddle point approximation}]{V\to\infty} Z \sim e^{-V\mathcal{H}(\phi_0)}\Big|_{\frac{\partial\mathcal{H}}{\partial\phi}|_{\phi_0}=0}$$

It's just Landau theory! So Landau theory can be derived from the general field theory, Landau-Ginzburg theory by mean field approximation.

From this example, we know that we can write down field theory only with symmetry. It helps us to get rid of any knowledge of microscopic details.

Next, consider 2-component real vector field

$$\vec{\phi} = (\phi_1, \phi_2)^T$$

Suppose that the theory is invariant under rotation of $\vec{\phi}$

$$\mathcal{H}[\phi(\vec{r})] = \int d\vec{r}\, [t|\vec{\phi}|^2 + u|\vec{\phi}|^4 + \underbrace{\frac{\alpha}{2}(\vec{\nabla}\vec{\phi})^2}_{\left(\sum_{i=1}^{d}\sum_{j=1}^{2}\partial_i\phi_j\right)^2} + \cdots]$$

It is called O(2)-model. The generalization to O(N)-model is directly. Landau-Ginzburg theory is the special case for $N=1$, Since $O(1) \simeq \mathbb{Z}_2$

We can also consider N-component complex vector field, then we obtain the U(N)-model. Similarly, N-component quaternion vector field gives us the Sp(N)-model.

And we can also consider field theory for a matrix field $Q_{n\times m}(\vec{r})$

$$\boxed{\mathcal{H}[Q(\vec{r})] = \frac{1}{2g}\int d\vec{r}\, \mathrm{Tr}\left[(\vec{\nabla} Q^{\dagger}(\vec{r}))\cdot(\vec{\nabla} Q(\vec{r}))\right]}$$

$Q(\vec{r}) \in U(N),\ O(N),\ Sp(N) \subset GL(N)$

It's the NLSM. (Non-Linear Sigma Model)

§2. Gaussian Theory.

We focus on the Landau-Ginzburg theory

$$H[\phi] = \int d^d\vec{r}\left[\frac{1}{2}(\vec{\nabla}\phi)^2 + t\phi^2 + u\phi^4\right]$$

Here, we assume $d=1$. Now, human being only know how to calculate the path integral explicitly for Gaussian Theory (i.e. $\mathcal{H}_0[\phi] = \frac{1}{2}(\vec{\nabla}\phi)^2 + t\phi^2$):

$$Z_0 = \int D\phi \exp\left[-\frac{1}{2}\int d^d\vec{r}\, \phi\, K\, \phi\right] \propto (\det K)^{-1/2},\quad K = -\partial^2 + 2t$$

This result is standard, and there are many different ways to derive it. Here we omit it. In statistical mechanics, we are interested in free energy

$$F = -\frac{1}{\beta}\log Z = \frac{T}{2}\log\det K$$

We need to solve the eigen-equation of K:

$$(-\vec{\nabla}^2 + 2t)\, e^{i\vec{k}\cdot\vec{r}} = (\vec{k}^2 + 2t)\, e^{i\vec{k}\cdot\vec{r}}$$

$$\Rightarrow \quad \det K = \prod_{\vec{k}\in\mathbb{R}^d} (\vec{k}^2+2t)$$

IR divergence

$$\Rightarrow \quad F = \frac{T}{2}\sum_{\vec{k}\in\mathbb{R}^d} \log(\vec{k}^2+2t) \overset{V\to\infty}{\rightsquigarrow} \frac{TV}{2}\int\frac{d^d\vec{k}}{(2\pi)^d}\log(\vec{k}^2+2t)$$

And the heat capacity can be calculated by

$$C_V = -T\frac{\partial^2}{\partial T^2}\frac{F}{V} \overset{t:=T-T_c}{\rightsquigarrow} -\frac{\partial^2}{\partial t^2}\frac{F}{V}$$

T_c is the critical temperature

$$\sim 2T_c\int\frac{d^d\vec{k}}{(2\pi)^d}\frac{1}{(\vec{k}^2+2t)^2} + \cdots$$

Leading Singularity.

Cutoff $\Lambda \sim \frac{1}{a}$

$$\sim \int_0^{\Lambda} k^{d-1}dk\,\frac{1}{(k^2+2t)^2}$$

$|\vec{k}|$

We are interested in the power-law divergence of C_V for $t\to 0$ (i.e. $T\to T_c$). To capture the t dependence, introduce $q := \frac{k}{\sqrt{t}}$, then:

$$C_V \sim t^{\frac{d}{2}-2}\int_0^{\frac{\Lambda}{\sqrt{t}}} dq\,\frac{q^{d-1}}{(q^2+2)^2}$$

Obviously, this integral only convergent for $1\le d<4$. In this case $C_V \sim t^{\frac{d}{2}-2}$. However, for the case $d>4$, the integral diverges, but $t^{\frac{d}{2}-2} \xrightarrow{t\to 0} 0$. Back to the previous expression and look more closely

$$C_V \propto \int_0^{\Lambda}\frac{k^{d-1}dk}{(k^2+2t)^2} \le \int_0^{\Lambda} k^{d-5}dk = \frac{\Lambda^{d-4}}{d-4} < \infty$$

So, Cutoff makes it finite and without t dependece. In Summary, for $d\ne 4$:

$$c \propto \begin{cases} t^{\frac{d}{2}-2} & (d<4) \quad \text{power divergence} \\ \underline{\text{finite const}} & (d>4) \end{cases}$$

↑ consist with mean field (Landau) theory

So, by considering the quantum fluctuation from QFT's view we predict the non-trivial behavior of c beyond the Landau theory in $d<4$ cases.

How about $d=4$? This case is subtle and related to logarithm divergence. Beyond the scope of this lecture. Next lecture will focus on the calculation of correlation functions and confirm $1/\sqrt{t}$ gives a length scale (i.e. it's a correlation length) explicitly

§3 Correlation Functions

We want to calculate the following objects:

$$\langle \phi(\vec{r}_1) \cdots \phi(\vec{r}_n) \rangle := \frac{1}{Z} \int \mathcal{D}\phi \, \phi(\vec{r}_1)\phi(\vec{r}_2)\cdots\phi(\vec{r}_n) e^{-H[\phi]}$$

Note: In statistical field theory, we are considering Euclidean Field Theory. So we don't have well-define time ordering (i.e. casual structure). Rather than defining n-pt function by $\langle 0 | T\{\phi(\vec{x}_1) \cdots \phi(\vec{x}_n)\} | 0 \rangle$ like relativistic QFT, here we just define it by path integral directly, which can be considered as analytic continues of $\langle 0|T\{\phi(t_1,\vec{r}_1)\cdots\phi(t_n,\vec{r}_n)\}|0\rangle$ by Wick rotation.

Using the Schwinger trick (i.e. add a source term)

$$Z \mapsto Z[h] := \int D\phi \, e^{-H[\phi] + \int d^d\vec{r}\, h(\vec{r})\phi(\vec{r})}$$

The physical meaning of $h(\vec{r})$ can be considered as $\vec{r}$-dependent magnetic field probe spontaneous breaking of $\mathbb{Z}_2$ spin flip symmetry. Mathmatically, it can be understood just as a trick to derive the n-pt function:

$$\langle\phi(\vec{r}_1)\cdots\phi(\vec{r}_n)\rangle = \frac{1}{Z}\left(\frac{\delta}{\delta h(\vec{r}_1)}\right)\cdots\left(\frac{\delta}{\delta h(\vec{r}_n)}\right)Z[h]\Big|_{h=0}$$

We mainly focus on the 2-pt function:

$$\langle\phi(\vec{r}_1)\,\phi(\vec{r}_2)\rangle = \frac{1}{Z}\frac{\delta^2 Z[h]}{\delta h(\vec{r})\delta h(\vec{r}')}\Big|_{h=0} = \frac{\delta\langle\phi(\vec{r})\rangle_h}{\delta h(\vec{r}')}\Big|_{h=0}$$

which is the magnetic susceptibility.

For Gaussian theory, $Z[h]$ can be calculated exactly

$$\frac{Z_0[h]}{Z_0[0]} = \exp\left[\frac{1}{2}\int d^d\vec{r}\, h(\vec{r})\left(-\nabla^2+2t\right)^{-1}h(\vec{r})\right]$$

Now we define the Green function:

$$G(\vec{r},\vec{r}') := \frac{\delta^2}{\delta h(\vec{r})\delta h(\vec{r}')}\log Z[h]\Big|_{h=0}$$

$$= -\frac{1}{Z^2}\frac{\delta Z}{\delta h(\vec{r})}\frac{\delta Z}{\delta h(\vec{r}')}\Big|_{h=0} + \frac{1}{Z}\frac{\delta Z}{\delta h(\vec{r})\delta(\vec{r}')}\Big|_{h=0}$$

$$= \langle\phi(\vec{r})\,\phi(\vec{r}')\rangle - \langle\phi(\vec{r})\rangle\langle\phi(\vec{r}')\rangle$$

$$= \langle\phi(\vec{r})\,\phi(\vec{r}')\rangle_c$$

So this is just the connected part of 2-pt function

$$Z[h] = Z[0]\,e^{W_c[h]}$$

dropped

$$\log Z[h] = \frac{1}{2}\int d^d\vec{r}\; h(\vec{r})K^{-1}h(\vec{r}) + \log Z[0]$$

$$= \frac{1}{2}\int d^d\vec{r}\int d^d\vec{r}'\; h(\vec{r})\left(\delta(\vec{r}-\vec{r}')\,K^{-1}\right)h(\vec{r}')$$

Since we only need $\frac{\delta}{\delta h}\log Z$, here we dropped the constant term out.

$$\langle\phi(\vec{r})\rangle_c = \frac{\delta}{\delta h(\vec{r})}\log Z[h]\Big|_{h=0} = K^{-1}h\Big|_{h=0} = 0$$

$$G(\vec{r},\vec{r}') = \langle\phi(\vec{r})\,\phi(\vec{r}')\rangle_c = \delta(\vec{r}-\vec{r}')\,K^{-1}$$

It means $G(\vec{r},\vec{r}')$ satisfies the following Green eq.

$$K\,G(\vec{r},\vec{r}') := (-\nabla^2 + 2t)\,G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

In QFT, it's called propagator. From EM theory, it's just the response of point charge. Using Fourier transformation, the Green eq. can be easily solved:

$G(\vec{r},\vec{r}')$ only depends on $\vec{r}-\vec{r}'$ by SO(3) symmetry.

$$G(\vec{r}) = \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\vec{k}^2+2t} e^{-i\vec{k}\cdot\vec{r}}$$

$\tilde{G}(\vec{k})$

$$= \frac{1}{(4\pi)^{d/2}} \int_0^\infty dx\, x^{-\frac{d}{2}} e^{-2tx-\frac{r^2}{4x}}$$

(Bessel function)

$$\underset{\text{approximation}}{\overset{\text{Saddle point}}{\sim}} \sqrt{\frac{\pi}{2S''(x_0)}}\, e^{-S(x_0)}, \quad S(x) := 2tx + \frac{r^2}{4x} + \frac{d}{2}\log x$$

$$S'(x_0)=0,\ x_0 = -\frac{d}{8t} + \sqrt{\left(\frac{d}{8t}\right)^2 + \frac{r^2}{8t}}$$

$$\overset{r \gg \frac{1}{\sqrt{t}}}{\sim} e^{-\sqrt{2t}\,r}$$

$\Rightarrow$ The two-point correlation function decays very quickly for $r \gtrsim \frac{1}{\sqrt{2t}}$, $\frac{1}{\sqrt{2t}}$ can be thought as a length scale, called correlation length.

On the other hand, the short distance behavior:

$$r \ll \frac{1}{\sqrt{t}} \Rightarrow x_0 \simeq \frac{r^2}{2d},\ S(x_0) \simeq \frac{d}{2}\log\frac{r^2}{2d},\ S''(x_0) \simeq \frac{2d^3}{r^4}$$

$$\Rightarrow \quad G(\vec{r}) \propto \frac{1}{r^{d-2}}$$

Generally, away the critical point

$$G(r) \propto e^{-\frac{r}{\xi}}, \quad \xi \propto \frac{1}{|t|^\nu}$$

At the critical point (i.e. $t \to 0$, $r \ll \frac{1}{\sqrt{t}} \to \infty$)

$$G(r) \propto \frac{1}{r^{d-2+\eta}}$$

η ← anomalous dimension

$d-2$ ↑ classical dimension

For Gaussian Theory:

$$\nu = \frac{1}{2}, \quad \eta = 0$$

§4 Renormalization Group

(这次课因为有事没去，故此笔记依据另一学生笔记作成)

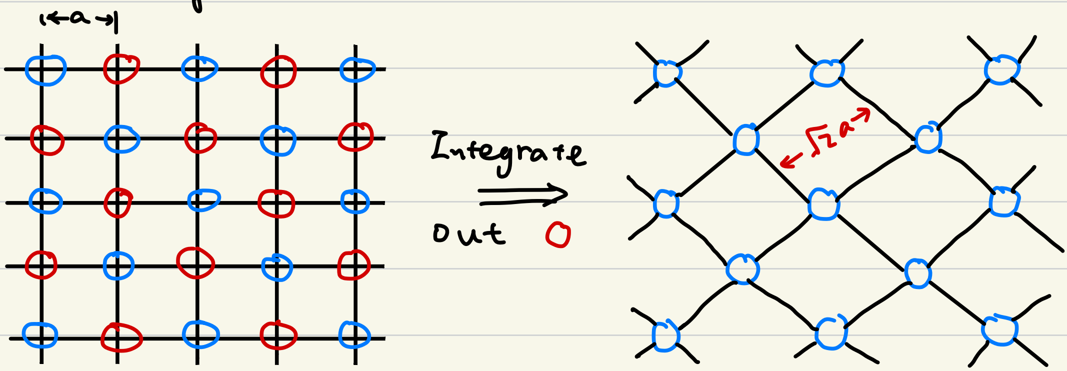
We are interested in capturing macroscopic (long distance) properties from microscopic (short distance) Models.

Renormalization is an efficient way to do that:

- Integrate out microscopic d.o.f.
- Rescale space(time) and fields

  ↑ in stastical physics only "space", no "time"

2D Ising Model



$$Z = \operatorname*{Tr}_{\{s\}} e^{-\mathcal{H}(\{s\})} = \operatorname*{Tr}_{\{s^{\color{red}\circ}\}\cup\{s^{\color{blue}\circ}\}} e^{-\mathcal{H}(\{s^{\color{red}\circ}\},\{s^{\color{blue}\circ}\})}$$

$$= \operatorname*{Tr}_{\{s^{\color{blue}\circ}\}} e^{-\mathcal{H}'(\{s^{\color{blue}\circ}\})}$$

The lattice constant is changed, so for a fair comparison, we need to rescale spae by

$$\vec{r} \longrightarrow \vec{r}\,' = \frac{\vec{r}}{\sqrt{2}}$$

This procedure can be done iteratively. Starting from $T > T_c$, where the correlation length is finite

$$\langle \phi(\vec{r})\,\phi(\vec{0})\rangle \propto e^{-r/\xi}$$

$$\xi \xrightarrow{RG} \frac{\xi}{b} \rightarrow \cdots \xrightarrow{(RG)^N} \frac{\xi}{b^N}, \quad N \to \infty \quad \xi^{(N)} \to 0$$

so after RG the macroscopic system will be less correlated and more disordered. Finally, we obtain a completely disordered phase corresponding to infinite temprature, since in Gaussian theory $\xi \propto \frac{1}{\sqrt{t}}$

On the other hand, we can also start from $T < T_c$

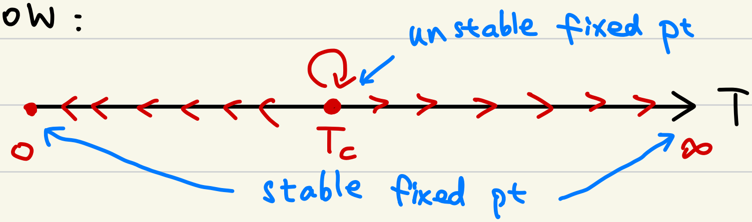
$$\langle \phi(r)\,\phi(\vec{0})\rangle \sim \langle \phi(\vec{0})\rangle^2 + (\text{something}) \cdot e^{-\frac{r}{\xi}} \xrightarrow{(RG)^N} \langle \phi(\vec{0})\rangle^2$$

We obtain a completely ordered phase at $T=0$

For the $\xi = \infty$ case i.e. at the critical point

$$\xi = \infty \xrightarrow{(RG)^N} \xi = \infty$$ The system is invariant

We can describe these three cases by the following RG flow:



Generic Hamiltonian has the following structure:

$$H = \underset{\uparrow\ \text{RG fixed pt}}{H^*} + \sum_n \underset{\uparrow\ \text{parameter}}{\lambda_n} H_n \leftarrow \text{purturbation (deformation)}$$

$\{H_n\}$ can be considered as a basis under RG flow.

$$\xrightarrow{RG} \quad H' = H^* + \sum_n \underbrace{\lambda_n'}_{= \lambda_n b^{y_n}} H_n$$

If $y_n > 0$, $\lambda_n \xrightarrow{RG} \infty$, H_n is RG-relevant

If $y_n < 0$, $\lambda_n \xrightarrow{RG} 0$, H_n is RG-irrelevant

In practically, most of the deformation to H^* are irrelevant, and there are only a few of relevant deformation. But, the critical phenomena can be described by H^* + (a few relevant deformation)

This is the "universality" in our nature i.e. Many different models are related to a single common RG fixed point

Symmetry generally constrains the manifold of RG flows. So symmetry controls the universality classes

We can also consider RG flow in momentum space. by Fourier transformation:

$$\phi(\vec{r}) = \int \frac{d^d \vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} \phi_{\vec{k}}$$

Low-$\vec{k}$: long-wave length (IR), slowly varying in real space

High-$\vec{k}$: Short-wave length (UV), rapidly oscillating in real space

In lattice model, we have a natural cutoff

$$|\vec{k}| < \Lambda \propto \frac{1}{a} \quad , \; a \text{ is the lattice constant}$$

This cutoff can be seen from the dicrete translation symmetry in lattice model. In momentum space k and $k+\frac{2\pi}{a}$ are identified, so k is in the Brilloun Zone

$$|k| \leq \frac{\pi}{a} \leftarrow \Lambda$$

§5. Momentum-Space Renormalization Group

Integrating out the high-momentum mode

$$\phi(\vec{r}) = \int_{|\vec{k}|<\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} \phi_{\vec{k}}$$

$$= \underbrace{\int_{|\vec{k}|<\frac{\Lambda}{b}} \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} \phi_{\vec{k}}}_{\text{long wave length } \phi^-_{\vec{k}}} + \underbrace{\int_{\frac{\Lambda}{b}<|\vec{k}|<\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{r}} \phi_{\vec{k}}}_{\text{short wave length } \phi^+_{\vec{k}}}$$

$$\Rightarrow \quad Z = \int \mathcal{D}\phi_{\vec{k}}\, e^{-H[\phi_{\vec{k}}]} = \int \mathcal{D}\phi^-_{\vec{k}} \mathcal{D}\phi^+_{\vec{k}}\, e^{-H[\phi^+_{\vec{k}}, \phi^-_{\vec{k}}]}$$

$$\longrightarrow \int \mathcal{D}\phi^-_{\vec{k}}\, e^{-H'[\phi^-_{\vec{k}}]}$$

$$\xrightarrow{\vec{k}\mapsto\vec{k}':=b\vec{k}} \int_{|\vec{k}'|<\Lambda} \mathcal{D}\phi_{\vec{k}'}\, e^{-H''[\phi_{\vec{k}'}]}$$

In the last step, we rescale the momentum to compare the renormalized Hamiltonian H'' with original Hamiltonian H.

$$H[\phi] = \underbrace{H^*[\phi]}_{\text{fixed pt.}} + \sum_{O}\int d^d\vec{r}\, g_O O(\vec{r})$$

The scale dimension of $O(\vec{r})$ is defined by rescaling $\vec{r}$ to $\vec{r}' = \frac{\vec{r}}{b}$, then

$$O(\vec{r}) \longmapsto O'(\vec{r}') = b^{\boxed{x_O}} O(\vec{r})$$

$x_O < d$: relevant , $x_O > d$: irrelevant

$$\int d^d\vec{r}'\, g_O' O' = \int d^d\vec{r}\, (b^{x_O-d} g_O')\, O$$

$$\longrightarrow \quad g_O \longmapsto g_O' = b^{d-x_O} g_O$$

Scaling dimensions provide critical exponents

Considering Landau-Ginzburg theory

$$\phi(\vec{r}) \longrightarrow \phi'(\vec{r}') = b^{x_\phi}\phi(\vec{r})$$

Hence we have the following scaling behavior of 2-pt function

$$\langle \phi(\vec{r})\phi(\vec{0})\rangle = \frac{1}{b^{2x_\phi}}\langle \phi'(\tfrac{\vec{r}}{b})\phi'(\vec{0})\rangle \qquad (t=t_c)$$

chose $b = |\vec{r}|$

$$\langle \phi(\vec{r})\phi(\vec{0})\rangle = \frac{1}{|r|^{2x_\phi}}\underbrace{\langle \phi'(\tfrac{\vec{r}}{|\vec{r}|})\phi'(0)\rangle}_{\text{const. since rotation invariance}} \propto \frac{1}{|r|^{2x_\phi}}$$

Comparing to $\langle \phi(\vec{r})\phi(\vec{0})\rangle \propto \frac{1}{|\vec{r}|^{d-2+\eta}}$

$$\Rightarrow \quad \boxed{X_\phi = \frac{d-2+\eta}{2}}$$

We can also do dimesional analyse for correlation length:

$$\xi \mapsto \xi' = \frac{\xi}{b}, \qquad t \mapsto t' = b^{y_t} t$$

$$\Rightarrow \xi t^{\frac{1}{y_t}} = \xi' t'^{\frac{1}{y_t}} = \text{Const} \Rightarrow \xi \propto t^{-\frac{1}{y_t}}$$

Comparing to $\xi \propto t^{-\nu} \Rightarrow \boxed{y_t = \frac{1}{\nu}}$

For Specific heat $C \propto |t|^{-\alpha}$, we consider the free energy (density) $f(t)$:

$$\int d^d\vec{r} f(t) = \int d^d\vec{r}' f(t') = \int d^d\vec{r}\left(b^{-d} f(b^{y_t} t)\right)$$

$$\Rightarrow f(t) = b^{-d} f(b^{y_t} t) \Rightarrow f(t) \propto t^{\frac{d}{y_t}}$$

$$C \propto \frac{\partial^2 f}{\partial t^2} \propto t^{\frac{d}{y_t}-2} \propto t^{-\alpha}$$

$$\Rightarrow \quad \boxed{\alpha = 2 - \frac{d}{y_t} = 2 - \nu d}$$

So there are non-trivial relations between different critical exponents, and these relations are model-independent, i.e. universality. Next we will see more and more relations

For field strength renormalization, $\phi \propto t^{\beta}$

$$\phi = b^{-X_\phi}\phi' = \left(\frac{t'}{t}\right)^{-\frac{X_\phi}{y_t}}\phi' \Rightarrow t^{-\frac{X_\phi}{y_t}}\phi = t'^{-\frac{X_\phi}{y_t}}\phi'$$

$$\Rightarrow \quad \phi \propto t^{\frac{x_\phi}{y_t}} \quad \Rightarrow \quad \boxed{\beta = \frac{x_\phi}{y_t} = \frac{(d-2+\eta)\nu}{2}}$$

Magnetic susceptibility $\chi \propto |t|^{-\gamma}$

$$\int d^d\vec{r}\, h\phi(\vec{r}) \overset{(RG)}{=} \int d^d\vec{r}'\, h'\phi'(\vec{r}')$$

$$= \int d^d\vec{r}\, (\underbrace{b^{x_\phi - d} h'}_{\text{〃}\, h})\, \phi(\vec{r})$$

$$\chi = \frac{\partial\phi}{\partial h} = \frac{\partial(b^{-x_\phi}\phi')}{\partial(b^{x_\phi - d} h')} = b^{d-2x_\phi}\chi' = \left(\frac{t'}{t}\right)^{\frac{d-2x_\phi}{y_t}}\chi'$$

$$\propto \left(\frac{1}{t}\right)^{\frac{d-2x_\phi}{y_t}}$$

$$\Rightarrow \quad \boxed{\gamma = \frac{d-2x_\phi}{y_t} = \nu(2-\eta)}$$

Finally, let's consider $\phi \propto |h|^{\frac{1}{\delta}}$

$$\phi = b^{-x_\phi}\phi' = \left(\frac{h}{h'}\right)^{\frac{x_\phi}{d-x_\phi}}\phi' \propto (h)^{\frac{x_\phi}{d-x_\phi}}$$

$$\Rightarrow \quad \boxed{\delta = \frac{d-x_\phi}{x_\phi} = \frac{d+2-\eta}{d-2+\eta}}$$

カイ セキ

以上は自明な次元解析でしょうか? Thinking about LG theory again:

$$H[\phi] = \int d^d\vec{r}\left[\frac{1}{2}(\vec{\nabla}\phi)^2 + \cdots\right]$$

$[\vec{r}] = L$, $[d^d\vec{r}] = L^d$, $[\vec{\nabla}] = L^{-1}$

$$L^0 = [H] = [d^d\vec{r}][\nabla^2][\phi^2] = L^{d-2}[\phi^2]$$

$$\Rightarrow \quad [\phi^2] = L^{-(d-2)} \quad \Rightarrow \langle \phi(\vec{r})\phi(\vec{0})\rangle \propto \frac{1}{r^{d-2}} \ ?$$

Gaussian theory indeed follows this naive dimensional analysis. However, it breaks down in generic theory Which

$$\langle \phi(\vec{r})\phi(\vec{0})\rangle \propto \frac{1}{r^{d-2+\eta}}$$

For interacting model, such as 2D/3D Ising $\eta \neq 0$ in general. By naive dimensional analysis we can also Consider

$$\int d^d\vec{r}\; t\phi^2 \rightarrow L^0 = [d^dr][t][\phi^2] = L^2[t]$$

$$\Rightarrow [t] = L^{-2} \Rightarrow \xi \propto |t|^{-\nu} \propto t^{-\frac{1}{2}} \Rightarrow \nu = \frac{1}{2} \ ?$$

However, this is also not true for generic theory.

So Why these naive dimensional analysis break down

So far, we have implicitly assumed that we only have a single length scale (r for $t=t_c$, ξ for $t\neq t_c$)

However, we always have an additional length scale

"Lattice spacing a"

In the presence of the two length scales, the dimensional analysis implies:

$$\langle \phi(\vec{r})\phi(\vec{0})\rangle \propto \frac{1}{r^{d-2}} f\left(\frac{a}{r}\right)$$

f is some function, so is a weaker constraint.

Since $a \ll r$, so f can behave as $f(x) \propto x^\eta$

$$\Rightarrow \langle \phi(\vec{r})\phi(\vec{0})\rangle \propto \frac{1}{r^{d-2}}\left(\frac{a}{r}\right)^\eta \propto \frac{1}{r^{d-2+\eta}}$$

η is called anomalous dimension denotes deviation from the naive dimensional analysis.

In hight dimension $(d>4)$ f is trivial, it's just mean-field (Gaussian) theory.

In lower dimension $(d<4)$ local fluctuations are significant makes f nontrivial leads to an anomalous dimension $\Rightarrow \eta \neq 0$. And this phenomena can be captured by RG transformations*

While the microscopic length of the lattice spacing is irrelevant to macroscopic physics, its existence is relevant

§6. Gaussian fixed point

Let us confirm that the Gaussian critical point is a fixed point of the RG flow.

$$H[\phi] = V\int_{|k|<\Lambda} \frac{d^d\vec{k}}{(2\pi)^d}\left(\frac{\vec{k}^2}{2}+t\right)|\vec{\phi}_k|^2$$

*: 4-dimension is subtle but is gaussian-like.

$$\Rightarrow Z = \int_{|\vec{k}|<\frac{\Lambda}{b}} \mathcal{D}\phi_{\vec{k}}\, e^{-H[\phi^-]} \underbrace{\int_{\frac{\Lambda}{b}<|\vec{k}|<\Lambda} \mathcal{D}\phi_{\vec{k}}\, e^{-H[\phi^+]}}_{\text{Const}}$$

$$\propto \int_{|\vec{k}|<\frac{\Lambda}{b}} \mathcal{D}\phi_{\vec{k}}\, e^{-H[\phi^-]}$$

In Gaussian theory we don't have the mixed term. So integrating out short-wave length modes is trivial

Next, we need to rescale momentum.

$$\vec{k} \longmapsto \vec{k}' := b\vec{k}$$

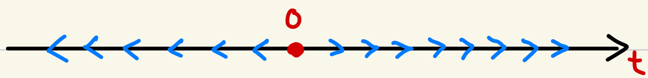
$$\Rightarrow Z = \int_{|\vec{k}'|<\Lambda} \mathcal{D}\phi_{\vec{k}'}\, e^{-H[\phi]}$$

Here, $H[\phi] = V\int \frac{d^d\vec{k}'}{(2\pi)^d}\left(\frac{(\vec{k}')^2}{2} + b^2 t\right) b^{-d-2}\, |\phi'_{\vec{k}'}|^2$

For $t=0$, i.e. critical point. $H[\phi]$ remains invariant if $\phi_{\vec{k}}$ scales by $\phi_{\vec{k}} \mapsto \phi'_{\vec{k}'} = b^{-\frac{d-2}{2}}\phi_{\vec{k}}$. So Gaussian critical point is indeed a fixed point under RG flow.

Since $t \mapsto t' = b^2 t$. At $t=0$, it's invariant

If we consider $t=\delta>0$, since $b^2>0$, so t becomes larger and large along the RG flow. Similarly, start from $t=-\delta<0$, it will be sent to $-\infty$.



So it's a unstable fixed point. In above, we focus on the momentum RG, next we will consider Real-space RG.

At Gaussian fixed point:

$$H[\phi] = \frac{1}{2}\int d^d\vec{r}\,(\nabla\vec{\phi})^2$$

Rescaling $\vec{r} \mapsto \vec{r}\,' = \frac{\vec{r}}{b}$

$$H[\phi] = \frac{1}{2}\int d^d\vec{r}\,'\,(\nabla'\vec{\phi})^2\, b^{d-2}$$

Hence, $H[\phi]$ remans invariant for $\phi \mapsto \phi' = b^{\frac{d-2}{2}}\phi$

For generic interaction

$$g_n\int d^d\vec{r}\,\phi^n = g_n\int b^d\, d^d\vec{r}\,\left(b^{-\frac{d-2}{2}}\phi'\right)^n$$

$$= \left(b^{(1-\frac{n}{2})d+n}\, g_n\right)\int d^d\vec{r}\,'\,(\vec{\phi}\,')^n$$

So the scaling dimension of g_n is

Gaussian $\quad y_n = (1-\frac{n}{2})d + n.$

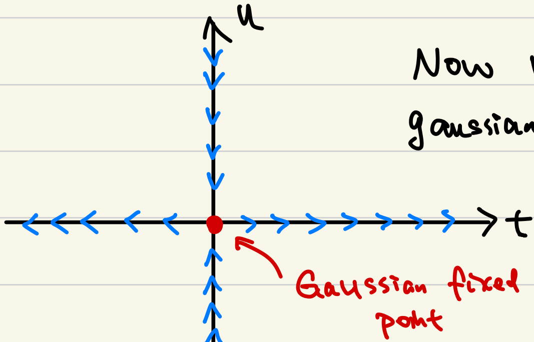
① $g_2 := t \Rightarrow y_2 = 2 > 0$ always relevant

② $g_4 := u \Rightarrow y_4 = -d+4$ $\begin{cases} \text{irrelevant} & d>4 \\ \text{relevant} & d<4 \end{cases}$

LG theory

③ $g_6 \Rightarrow y_6 = 6-2d$ $\begin{cases} \text{relevant for } d<3 \\ \text{irrelevant for } d>3 \end{cases}$

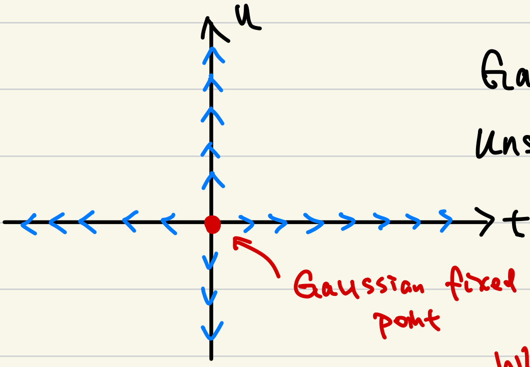
$d>4$



Now with 2 parameter, gaussian fixed point is stable along u, so it describes the Ising critical point for $d>4$

$t=0$ is the interface of order and disorder phase

$d<4$



Gaussian fixed point becomes unstable for $d<4$

We need to finetune u and t as 0

What we do in experiment

However, we only have to finetune one parameter t to reach the Ising Critical point.* So Gaussian fixed point does not describe the Ising critical point for $d<4$. It means an additional fixed point should appear for $d<4$. We need to perform RG transformation in the presence of interactions.

In $d=2$ dimensions, $\phi'(\vec{r}') = \phi(\vec{r}) \Rightarrow y_n = 2 > 0$ for any n. All of ϕ^n's are relevant infinite number of relevant perturbations. There are infinite number of nontrivial fixed points

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* If there are no relevant direction, it means we obtain a stable phase, but here we focus on phase transition, temperature t control the phase and at the critical point, we obtain a universal fixed point

Emergent Symmetry

When we write down the Landau-Ginzburg theory we implicitly incorporate symmetry that is not respected in the corresponding lattice model. This is justified by confirming many perturbations are irrelevant.

For example, 2D Ising model on a square lattice. C_4 rotation symmetry is respected but continuous rotation symmetry $SO(2)$ is NOT respected. Consider the following operator $\mathcal{O} := \phi \partial_x^4 \phi + \phi \partial_y^4 \phi$, which preserve C_4 rotation symmetry, but $SO(2)$ symmetry is broken.

However, scaling dimension of $\mathcal{O}$ is $-2 < 0$. It's irrelevant. Therefore, continuous rotation symmetry emerges at a critical point. But this happen accidentally, we don't have any general prove.

§7. Perturbative Renormalization Group

For Landau-Ginzburg theory, we knew that t and u has the following rescaling dimension:

$$\vec{r} \longmapsto \vec{r}' = \frac{\vec{r}}{b} \Rightarrow \begin{cases} t \mapsto t' = b^2 t \\ u \mapsto u' = b^{4-d} u \end{cases}$$

Untile now we have not constrained how large b is.

Actually, we are interested in the infinitesimal rescaling, which means $b = 1 + dl$, $dl \ll 1$. Then we have $t' = (1+dl)^2 t = (1+2dl)t + \mathcal{O}(dl^2)$. We can define β-function as:

$$\frac{dt}{dl} = 2t := \beta_t(t)$$

As well as $\beta_u(u)$:

$$\frac{du}{dl} = (4-d)u := \beta_u(u)$$

Obviously, fixed points are given by zeros of β-fuctions, i.e.

$$\left.\frac{du}{dl}\right|_{u=u^*} = \left.\frac{dt}{dl}\right|_{t=t^*} = 0$$

But rescaling only captures the leading order contributions of RG. Indeed, this is only the first step of RG, we still need to do the second step, coarse grainning to work out the anomaly dimensions. For free theory, i.e. Gaussian theory, this second step is trivial, but it's non-trivial in interacting QFT. Coarse graining is very important, it will give rise to a new nontrivial fixed point.

$$H[\phi] = H^*[\phi] + g_i \int d^d\vec{r}\ O_i(\vec{r})$$

$$Z^* := \int \mathcal{D}\phi \, e^{-H^*[\phi]} , \quad \langle \Phi \rangle := \frac{1}{Z^*}\int \mathcal{D}\phi \, e^{-H^*[\phi]} \Phi$$

$$\Rightarrow Z = Z^*\Big(1 - \sum_i g_i \int d^d\vec{r} \langle O_i(\vec{r})\rangle + \frac{1}{2}\sum_{ij} g_i g_j \int d^d\vec{r}_1 \, d^d\vec{r}_2 \langle O_i(\vec{r}_1) O_j(\vec{r}_2)\rangle\Big)$$

- $O(g_i)$

$$g_i \int d^d\vec{r} \langle O_i(\vec{r})\rangle = g_i \int (b^d d^d\vec{r}\,') \langle b^{-x_i} O_i'(\vec{r}\,')\rangle$$

$$= \underbrace{(b^{d-x_i} g_i)}_{g_i'} \int d^d\vec{r}\,' \langle O_i'(\vec{r}\,')\rangle$$

$$\Rightarrow g_i' = b^{d-x_i} g_i = (1+dl)^{d-x_i} g_i = (1+(d-x_i)dl) g_i$$

$$\Rightarrow \frac{dg_i}{dl} = (d-x_i) g_i + O(g^2)$$

- $O(g_i g_j)$

$\vec{r}_1$ cannot approach $\vec{r}_2$ within the lattice spacing

i.e. we have the short-distance cut off a

$$g_i g_j \int_{|\vec{r}_1-\vec{r}_2|>a} d^d\vec{r}_1 \, d^d\vec{r}_2 \langle O_i(\vec{r}_1) O_j(\vec{r}_2)\rangle$$

$$\overset{\text{rescale}}{=} g_i g_j \int_{|\vec{r}_1\,'-\vec{r}_2\,'|>\frac{a}{b}} b^{2d} \, d^d\vec{r}_1\,' d^d\vec{r}_2\,' \langle O_i'(\vec{r}_1\,') O_j'(\vec{r}_2\,')\rangle \cdot b^{-x_i-x_j}$$

$$= g_i g_j \, b^{2d-x_i-x_j} \Big[\int_{\frac{a}{b}<|\vec{r}_1\,'-\vec{r}_2\,'|<a} d^d\vec{r}_1\,' d^d\vec{r}_2\,' \langle O_i'(\vec{r}_1\,') O_j'(\vec{r}_2\,')\rangle$$

$$+ \int_{|\vec{r}_1' - \vec{r}_2'| > a} d^d\vec{r}_1' d^d\vec{r}_2' \langle O_i'(\vec{r}_1') O_j'(\vec{r}_2') \rangle \Big]$$

The first contribution is new, to evaluate it, since $b \sim 1$, $O(\vec{r}_1) O(\vec{r}_2)$ looks like a single operator at $\frac{\vec{r}_1 + \vec{r}_2}{2}$, we can use OPE

$$O_i(\vec{r}_1) O_j(\vec{r}_2) \sim \sum_k C_{ijk}(\vec{r}_1 - \vec{r}_2) O_k\left(\frac{\vec{r}_1 + \vec{r}_2}{2}\right)$$

$$b^{-x_i - x_j} O_i'(\vec{r}_1') O_j'(\vec{r}_2') \qquad b^{-x_k} O_k'$$

$$\Rightarrow \quad C_{ijk}(\vec{r}_1 - \vec{r}_2) = \frac{C_{ijk}(\vec{r}_1' - \vec{r}_2')}{b^{x_i + x_j - x_k}} \xlongequal{\text{Conformal}} \frac{C_{ijk}}{|\vec{r}_1 - \vec{r}_2|^{x_i + x_j - x_k}}$$

$$\int_{\frac{a}{b} < |\vec{r}_1' - \vec{r}_2'| < a} d^d\vec{r}_1' d^d\vec{r}_2' \langle O_i'(\vec{r}_1') O_j'(\vec{r}_2') \rangle \quad \text{(for simplicity, } a=1\text{)}$$

$$= \sum_k \int_{(1-dl) < |\vec{r}_1' - \vec{r}_2'| < 1} d^d\vec{r}_1' d^d\vec{r}_2' \frac{b^{x_i + x_j - x_k} C_{ijk}}{|\vec{r}_1 - \vec{r}_2|^{x_i + x_j - x_k}} \left\langle O_k'\left(\frac{\vec{r}_1' + \vec{r}_2'}{2}\right)\right\rangle$$

$$\simeq \sum_k \int_{1-dl < |\delta| < 1} d^d\delta' d^d\vec{r}' \frac{C_{ijk}}{|\delta'|^{x_i + x_j - x_k}} \langle O_k'(\vec{r}') \rangle$$

Here, we defined $\delta' := \vec{r}_1 - \vec{r}_2'$, $\vec{r}' := \frac{\vec{r}_1' + \vec{r}_2'}{2}$

Area of $d-1$-sphere

$$= S^{d-1} \cdot dl \sum_k \int d^d\vec{r}' \langle O_k'(\vec{r}') \rangle C_{ijk}$$

$$\Rightarrow Z = Z^* \Big(1 - \sum_i g_i \int d^d\vec{r} \langle O_i(\vec{r}) \rangle + \frac{1}{2} \sum_{ij} g_i g_j \int d\vec{r}_1 d\vec{r}_2$$

$$\langle O_i(\vec r_1)\, O_j(\vec r_2)\rangle \Big)$$

$$= Z^* \Big[1 - \sum_k \Big(\overbrace{\underbrace{b^{d-x_k}}_{1+(d-x_k)\cdot dl} g_k - \sum_{ij} \frac{g_i g_j}{2} S^{d-1} \cdot dl\, C_{ijk}}^{g_k'} \Big)$$

$$\times \int d^d \vec r' \langle O_k'(\vec r')\rangle + \frac{1}{2}\sum_{ij} g_i g_j\, b^{2d-x_i-x_j}$$

will correct

$$\times \int_{|\vec r_1' - \vec r_2'|>a} d^d \vec r_1'\, d^d \vec r_2' \langle O_i'(\vec r_1')\, O_j'(\vec r_2')\rangle + \cdots \Big]$$

$$\Rightarrow dg_k = (d-x_k)\, dl \cdot g_k - \sum_{ij} \frac{g_i g_j}{2} S^{d-1} \cdot dl\, C_{ijk}$$

$$\Rightarrow \boxed{\begin{array}{l} \beta_k := \dfrac{d\tilde g_k}{dl} = (d-x_k)\,\tilde g_k - \sum_{ij} C_{ijk}\, \tilde g_i\, \tilde g_j + O(g^3) \\ \text{Here}, \quad \tilde g_i := \dfrac{S_{d-1}}{2} g_i \end{array}}$$

§8 Operator product Expansion (这次在别的设备上写的，风格不太一样)

3-pt Correlation function Can be represented by x_i and C_{ijk}

$$\langle \overbrace{O_i(\vec r_1)\; O_j(\vec r_2)}^{\text{OPE}}\; O_k(\vec r_3)\rangle = \sum_{k'} \frac{C_{ijk'}}{|\vec r_1 - \vec r_2|^{x_i + x_j - x_{k'}}} \Big\langle O_{k'}\Big(\frac{\vec r_1 + \vec r_2}{2}\Big)\, O_k(\vec r_3)\Big\rangle$$

$$= \frac{C_{ijk}}{|\vec r_1 - \vec r_2|^{x_i+x_j-x_k} \left|\frac{\vec r_1+\vec r_2}{2} - \vec r_3\right|^{2x_k}}$$

Here we used the following normalization

$$\langle O_i(\vec r_1)\, O_j(\vec r_2)\rangle = \frac{\delta_{ij}}{|\vec r_1 - \vec r_2|^{2x_i}}$$

In general, OPE coefficients C_{ijk} are difficult to find.

However, for the Gaussian fixed point, we can obtain C_{ijk}'s via simple combinatorics thanks to Wick's theorem

Gaussian fixed point, 2-pt correlation function

$$\langle \phi(\vec{r}_1)\,\phi(\vec{r}_2)\rangle = \frac{1}{|\vec{r}_1-\vec{r}_2|^{d-2}}$$

For example. Let's consider $\phi(\vec{r}_1)^2\phi(\vec{r}_2)^2$

$$\langle \phi(\vec{r}_1)^2\phi(\vec{r}_2)^2\, O(\vec{r})\rangle \overset{|\vec{r}-\vec{r}_{1,2}|\gg|\vec{r}_1-\vec{r}_2|}{\sim} \langle \phi\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)^4 O(\vec{r})\rangle$$

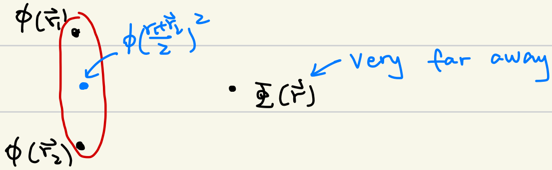
$$+\,4\langle \phi(\vec{r}_1)\,\phi(\vec{r}_2)\rangle\langle \phi\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)^2\Phi(\vec{r})\rangle$$

$$+\,2\langle \phi(\vec{r}_1)\,\phi(\vec{r}_2)\rangle^2\langle \Phi(\vec{r})\rangle$$

Here, we use the fact

$$\langle :\phi(\vec{r}_1)\,\phi(\vec{r}_2):\, O(\vec{r})\rangle \overset{|\vec{r}-\vec{r}_{1,2}|\gg|\vec{r}_1-\vec{r}_2|}{\sim} \langle \phi\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)^2\Phi(\vec{r})\rangle$$

Diagramatically, it means



On the other hand, we can represent this correlator by using OPE.

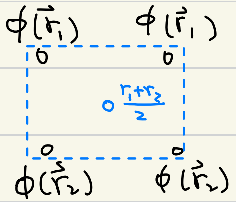
$$\langle \phi(\vec{r}_1)^2\phi(\vec{r}_2)^2\, O(\vec{r})\rangle \sim \sum_k \frac{C_{ijk}}{|\vec{r}_1-\vec{r}_2|^{2d-4-x_k}}\langle \Phi_k\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right) O(\vec{r})\rangle$$

Comparing these two equations, it's easy to obtain

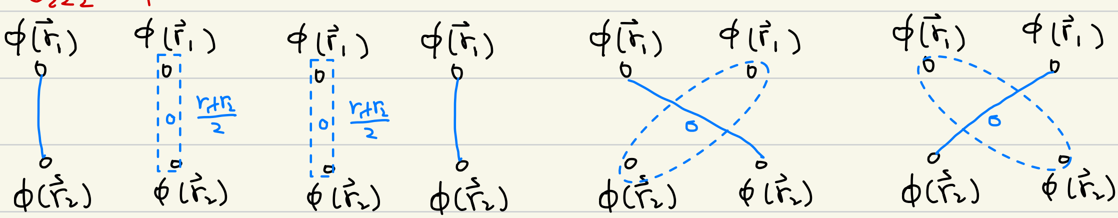
$$\phi(\vec r_1)^2\,\phi(\vec r_2)^2 \sim \overset{C_{224}}{1}\,\phi\left(\frac{\vec r_1+\vec r_2}{2}\right)^4 + \frac{\overset{C_{222}}{4}}{|\vec r_1-\vec r_2|^{d-2}}\,\phi\left(\frac{\vec r_1+\vec r_2}{2}\right)^2 + \frac{\overset{C_{220}}{2}}{|\vec r_1-\vec r_2|^{d-4}}$$

This result can be collected as diagrams

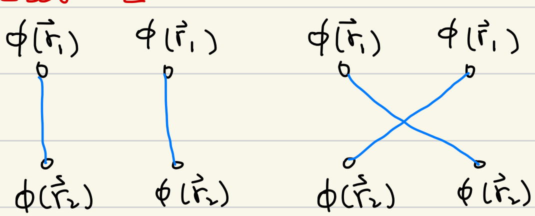
- $C_{224}=1$



- $C_{222}=4$



- $C_{220}=2$



Implicitly, we defined $\phi(\vec r)^2$ by normal ordering $:\phi(\vec r)^2:$ such that $\langle :\phi(\vec r)^2: \rangle \sim 0$ has been excluded. Actually, $\phi(\vec r)^2$ itself is NOT well-define since

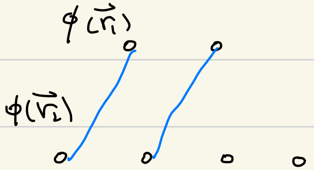
$$\langle \phi(\vec{r})\phi(\vec{r})\rangle = \frac{1}{|\vec{r}-\vec{r}|^{d-2}} \sim \infty$$

In the presence of the short-distance cutoff, it should be understood as a^{-d+2} In the continuum limit, we introduce Nop to avoid such divergence. We can safely work with it since we are focus on RG equations which is not sensitive on that

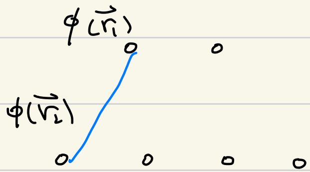
Now consider a more complicated example

$$\phi^2 \phi^4 \sim \frac{C_{242}}{r^{d-2}}\phi^2 + C_{244}\phi^4 + C_{246}\, r^{d-2}\phi^6$$

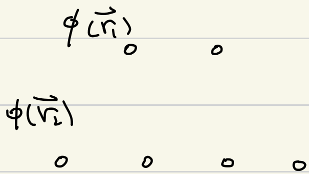
We can calculate coefficients by diagrams as before.



$$C_{242} = A_4^2 = \frac{4!}{2!} = 12$$



$$C_{244} = 2C_4^1 = 8$$



$$C_{246} = C_4^0 = 1$$

Here, for simplicity, I only draw one diagram for each C_{ijk}

Now, we have enough inpute for the β function, plug them into β_u, β_t

$$\beta_t(t,u) = 2t - C_{222}t^2 - (C_{242}+C_{422})\,tu - C_{442}\,u^2$$
$$\beta_u(t,u) = (4-d)u - C_{224}\,t^2 - 2(C_{244}+C_{422})\,tu - C_{444}\,u^2$$

OPE is associative and commutative, so

$$C_{ijk} = C_{jik}$$

And $C_{442} = A_4^3 \times C_4^3 = 96$

$C_{444} = A_4^2 \times C_4^2 = 72$

RG fixed point satisfies $\beta_t = \beta_u = 0$. obviously, $u = t = 0$ is a trivial solution, i.e. Gaussian fixed point. We seek an additional fixed point for $d<4$, Let's consider $d = 4-\varepsilon$, $\varepsilon > 0$

$$\beta_t = 2t - 4t^2 - 24tu + 96u^2 = 0$$

impose $t_*, u_* = O(\varepsilon)$ $\Rightarrow$ $t_* = O(\varepsilon^2)$

$$\beta_u = \varepsilon u - t^2 - 16tu - 72u^2 = 0$$

Again, impose $t_*, u_* = O(\varepsilon)$ $\Rightarrow$ $u_* = 0$ or $\frac{\varepsilon}{72} + O(\varepsilon^2)$

So we find a new fixed point, Wilson-Fisher fixed point:

$$(t_*, u_*) = \left(0, \frac{\varepsilon}{72}\right) + O(\varepsilon^2)$$

§9 Perturbative RG

Let's linearize the RG equations around the Wilson-Fisher fixed point.

$$\frac{dt}{dl} \simeq \frac{\partial \beta_t}{\partial t}(t_*, u_*)\,t + \frac{\partial \beta_t}{\partial u}(t_*, u_*)\,u$$

$$= (2 - 8t_* - 24u_*)t + (-24t_* - 192u_*)u$$
$$= \left(2 - \frac{\varepsilon}{3}\right)t - \frac{8\varepsilon}{3}u$$

As well as u:

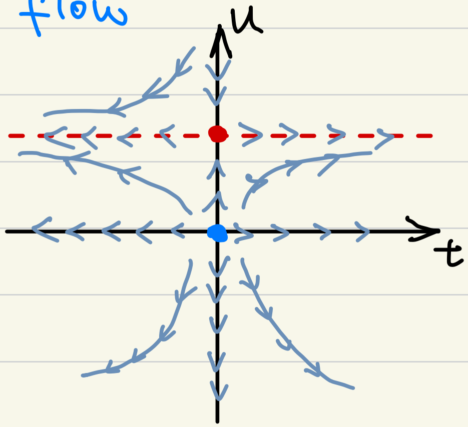
$$\frac{du}{dl} = \frac{\partial \beta_u}{\partial t} \cdot t + \frac{\partial \beta_u}{\partial u} \cdot u = -\frac{2\varepsilon}{9}t - \varepsilon u$$

$$\Rightarrow \frac{d}{dl}\begin{pmatrix} t \\ u \end{pmatrix} \simeq \begin{pmatrix} 2 - \frac{\varepsilon}{3} & -\frac{8\varepsilon}{3} \\ -\frac{2\varepsilon}{9} & -\varepsilon \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix}$$

The scaling dimension of t and u is given by the eigenvalues of the matrix of coefficients

$$y_t = 2 - \frac{\varepsilon}{3}, \quad y_u = -\varepsilon$$

RG flow



● Wilson-Fisher fixed point

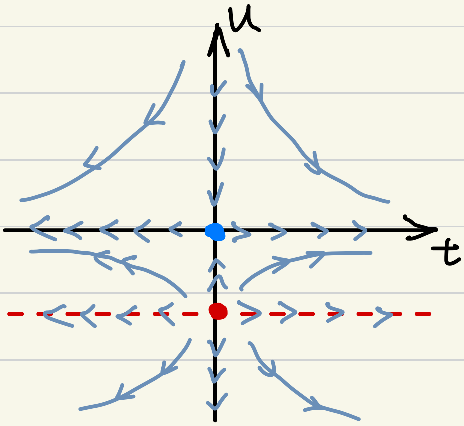
● Gaussian fixed point

$\varepsilon > 0$ $(d < 4)$

The Wilson-Fisher fixed point gives you the Ising critical point

Then turn to the $\varepsilon < 0$ case, i.e. $d = 4 - \varepsilon > 4$.

The RG flow is given by:



• Wilson-Fisher fixed point

• Gaussian fixed point

$\varepsilon > 0$ $(d<4)$

It means only gaussian fixed point is stable (for u)

We can also calculate the corrected critical exponent

$$\nu := \frac{1}{y_t} = \frac{1}{2} + \frac{\varepsilon}{12}$$

plug $\varepsilon = 1$ in (although it doesn't satisfy ($\varepsilon \ll 1$)

$$\nu = \frac{7}{12} = 0.5833\cdots$$

Now it's closer to the numerical result $0.6299\cdots$

In principle, we can improve this result by higher-order calcultation. Rather, the significance of Wilson-Fisher RG is to analytically demonstrate the emergence of non-trivial fixed points

Now, we will give some comments on 4-dimension

In $d=4$ dimensions, the Wilson-Fisher fixed point coincides with the Gaussian fixed point. Correspondingly, the scaling dimension for u vanishes:

$$y_u = 0 \quad , \text{ marginal.}$$

Back to the RG equation:

$$\left.\frac{du}{dl}\right|_{\varepsilon=0} = -72u^2 \Rightarrow u(l) = \frac{1}{u_0^{-1}+72l}$$

Here $u_0 := u(l=0)$, similarly:

$$\left.\frac{dt}{dl}\right|_{\varepsilon=0} = 2(1-12u)t = 2\left(1-\frac{12}{u_0^{-1}+72l}\right)t$$

$$\Rightarrow \log\frac{t}{t_0} = 2l - \frac{1}{3}\log(72u_0 l+1), \quad t_0 := t(l=0)$$

Gaussian contribution

$$\Rightarrow e^l = \left(\frac{t}{t_0}\right)^{\frac{1}{2}}(72u_0 l+1)^{\frac{1}{6}}$$

$$\underbrace{e^l \approx \left(\frac{t}{t_0}\right)^{\frac{1}{2}}} \left(\frac{t}{t_0}\right)^{\frac{1}{2}}\left(36u_0\log\frac{t}{t_0}+1\right)^{\frac{1}{6}}$$

$$\propto t_0^{-\frac{1}{2}}(\log t_0)^{\frac{1}{6}}$$

Recall that

$$\vec{r} \mapsto \vec{r}' = \frac{\vec{r}}{b} := \frac{\vec{r}}{1+dl} = (1-dl)\vec{r} \Rightarrow \frac{r}{r_0} = e^{-l}$$

r_0 is the scale of the system (l is just a pseudo parameter)

So we obtain the behaviour of correlation length

$$r_0 \propto e^l \propto \underbrace{t_0^{-\frac{1}{2}}}_{\text{Gaussian}} \underbrace{(\log t_0)^{\frac{1}{6}}}_{\text{logarithmic correction}}$$

Actually the power $\frac{1}{6}$ comes from $\frac{C_{242}}{C_{444}} = \frac{12}{72}$, so it is universal.

So far we implicitly assume the perturbative expansion well-define. Is it true? Back to the partition function.

$$Z = \int \mathcal{D}\phi \, e^{-H[\phi]}, \quad H[\phi] = \int d^d \vec{r} \left[-\frac{1}{2}(\nabla\phi)^2 + t\phi^2 + u\phi^4 \right]$$

Actually, Z is not analytic around $u=0$, since $Z<\infty$ if $u>0$ and $Z=\infty$ if $u<0$. This means perturbative expansion for small u is NOT convergent, however, it can be considered as an asymptotic expansion.

We can see that from the baby version, 0d

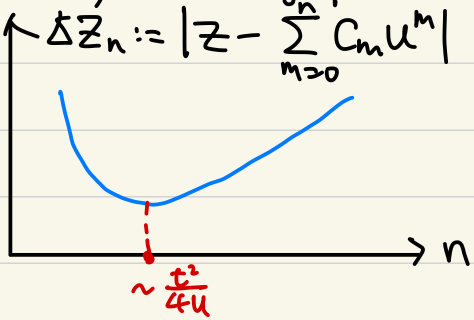
$$Z = \sqrt{\frac{t}{\pi}} \int_{-\infty}^{+\infty} d\phi \, e^{-t\phi^2 - u\phi^4} \quad (t>0, u>0)$$

$$= \sqrt{\frac{t}{\pi}} \int_{-\infty}^{+\infty} d\phi \, e^{-t\phi^2} \cdot \sum_{n=0}^{\infty} \frac{(-u\phi^4)^n}{n!}$$

$$= \sum_n C_n u^n, \quad C_n \overset{n\to\infty}{\sim} \frac{1}{\sqrt{\pi n}} \left(-\frac{4n}{e t^2} \right)^n$$

$|C_n u^n|$ diverges for $n \to \infty$ for arbitrary u.

However, the asymptotic expansion is still useful



It means the accuracy first increase, then decrease for higher and higher order result

In practically, human being can only calculate first few order contributions. But the turning point is about $\frac{t^2}{4u} \sim 25$ for $u=0.01$, $t=1$. So we don't need to worry about it,

§10 Continuous Symmetry. (仕事があるので、欠席(けっせき)した)

So far we have focus on the Ising model. It has $\mathbb{Z}_2$ symmetry. Continuous symmetry leads to further rich many-body phenomena. In this lecture, we will back to the O(N)-model, N=1 is just the Ising field theory.

$$H_{O(N)}[\vec{\phi}] = \int d^d\vec{r} \left[\frac{1}{2}(\vec{\nabla}\vec{\phi})^2 + t|\vec{\phi}|^2 + u|\vec{\phi}|^4 \right], \quad \vec{\phi} \in \mathbb{R}^N$$

It corresponds to the following lattice model

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j, \quad |\vec{S}_i| = 1$$

N=1, Ising model

N=2, XY model

N=3, Heisenberg model.

For O(N)-model we have the following perturbative RG equations:

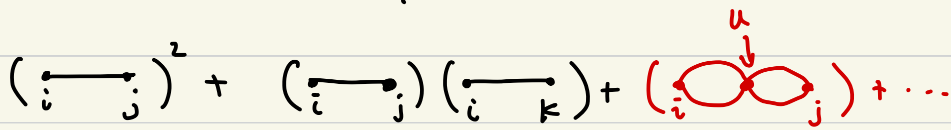
$$\frac{dt}{dl} \simeq 2t - C_{242}tu \qquad \frac{du}{dt} \simeq (4-d)u - C_{424}u^2$$

We first compute C_{242}:

$$|\vec{\phi}|^2 \cdot |\vec{\phi}|^4 \sim C_{242}|\vec{\phi}|^2 + \cdots$$

$$\text{LHS} \Rightarrow \langle |\vec{\phi}|^2 \cdot |\vec{\phi}|^4 \cdot O \rangle = \sum_{i,j,k} \langle \phi_i(\vec{r}_1)^2 \phi_j(\vec{r}_2)^2 \phi_k(\vec{r}_2)^2 O(\vec{r}) \rangle$$

To obtain $|\vec{\phi}|^2 = \sum_k \phi_k(\vec{r}_2)^2$ we have the following diagrams



Noting now O(N) is not a free theory, it has 4-vertex. However it only contributes to loop-level. we only need to consider the first two diagrams, which give us:

$$\sum_{i,j,k} \langle \phi_i(\vec{r}_i)^2 \phi_j(\vec{r}_j)^2 \phi_k(\vec{r}_k)^2 O(\vec{r}) \rangle$$

$$\sim 2 \sum_{ijk} \langle \phi_i(\vec{r}_i) \phi_j(\vec{r}_j) \rangle^2 \langle \phi_k(\vec{r}_k)^2 O(\vec{r}) \rangle + (j \leftrightarrow k)$$

$$+ 2^3 \sum_{ijk} \langle \phi_i(\vec{r}_i) \phi_j(\vec{r}_j) \rangle \langle \phi_i(\vec{r}_i) \phi_k(\vec{r}_k) \rangle \langle \phi_k(\vec{r}_k) \phi_j(\vec{r}_j) O(\vec{r}) \rangle$$

$$\underset{\langle \phi_i \phi_j \rangle = \delta_{ij}}{\sim} 2 \times 2N \langle |\phi|^2 O \rangle + 2^3 \langle |\phi|^2 O \rangle$$

Then we have $C_{242} = 4N + 8$, Similarly, one can find

$$C_{444} = 8N + 64$$

plug into the RG equations:

$$\frac{du}{dl} \simeq (\varepsilon - 8(N+8)u)u \rightarrow u^* = \frac{\varepsilon}{8(N+8)}$$

$$\Rightarrow \frac{dt}{dl} \simeq 2(1 - 4(N+2)u)t \simeq \underbrace{2\left(1 - 4(N+2)\frac{\varepsilon}{8(N+8)}\right)}_{y_t} t$$

$\Rightarrow$ critical exponent

$$\nu = \frac{1}{y_t} = \frac{1}{2} + \frac{N+2}{4(N+8)} \varepsilon$$

It's depend on N, so $O(N)$ has different universality

Take $N \to \infty \Rightarrow u^* = O(\frac{1}{N}) \quad y_t = d-2$

which suggets that the perturbative analysis becomes exact. It can be confirms by large N expansion.

Next, we explain gapless excitation, Nambu-Goldstone modes when SSB.

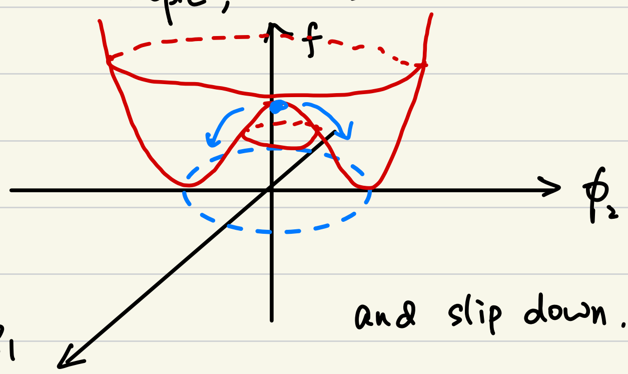
From the mean-field approximation

$$f = t|\vec{\phi}|^2 + u|\vec{\phi}|^4 = u\left(|\vec{\phi}|^2 + \frac{t}{2u}\right)^2 - \frac{t^2}{4u} \quad (u>0)$$

$$\Rightarrow \begin{cases} t>0 & : \langle|\vec{\phi}|^2\rangle = 0 \\ t<0 & : \langle|\vec{\phi}|^2\rangle = \sqrt{-\frac{t}{2u}} > 0 \end{cases}$$

there is a phase transition at $t=0$. However, now $\vec{\phi}$ is a N-components vector, $\langle|\phi|^2\rangle$ doesn't determin the direction of $\vec{\phi}$

For exanple, $N=2$

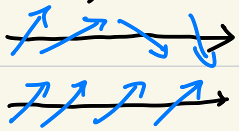


When $t<0$, the GS $\langle|\vec{\phi}|^2\rangle = 0$ becomes semi-stable, it will chose a special direction and slip down. $O(N)$ is broken spontaneously

You may think we can also make the direction of $\vec{\phi}$

Varies, since it gives you the same mean-field theory.

energy gap Δf



but it actually has higher energy due to the gradient term $\frac{1}{2}(\nabla\vec{\phi})^2$

$$\Delta f \sim \frac{1}{2}\left(\frac{\alpha}{L}\right)^2 \quad , \ \alpha : \text{entire change of direction}$$

It can be arbitrarily small for a large system ($L \to \infty$)

Then it can be consider as a gapless excitation, i.e Nambu-Goldstone modes.

We can also check it by explicitly expand around the grond states.

① $N=2$ (XY model)

$$\vec{\phi}(\vec{r}) \simeq \sqrt{-\frac{t}{2u}} \begin{pmatrix} \cos\theta(\vec{r}) \\ \sin\theta(\vec{r}) \end{pmatrix} \quad t<0,\ u>0$$

$$\Rightarrow H[\phi] \simeq \frac{1}{2}\left(-\frac{t}{2u}\right)\int d^d\vec{r}\,(\vec{\nabla}\theta)^2 + \underline{\text{const}}$$

↖ no mass term

② $N=3$ (Heisenberg model

$$\vec{\phi}(\vec{r}) \simeq \sqrt{-\frac{t}{2u}} \begin{pmatrix} \sin\theta(\vec{r})\cos\phi(\vec{r}) \\ \sin\theta(\vec{r})\sin\phi(\vec{r}) \\ \cos\theta(\vec{r}) \end{pmatrix}$$

$$\Rightarrow H[\vec{\phi}] = \frac{1}{2}\left(-\frac{t}{2u}\right)\int d^d\vec{r}\,\left[(\vec{\nabla}\theta)^2 + \sin^2\theta(\nabla\phi)^2\right] + \text{const}$$

Here we have two interacting NG modes but the mass terms are missed.

However, In the above analysis, we only consider the fluctuation of direction of $\vec{\phi}$, we can also consider the fluctuation of amplitude of $\vec{\phi}$ (i.e. $|\vec{\phi}|$), which gives us the Higgs modes.

In the end, we conclude that the number of NG modes can be considered as a coset of symmetry groups:

$$S^{N-1} \simeq \frac{O(N)}{O(N-1)} \begin{matrix} \leftarrow \text{before SSB} \\ \leftarrow \text{after SSB} \end{matrix}$$

$\#$ NG modes $= \dim S^{N-1} = N-1$ for $O(N)$-model

§11 Hohenberg-Mermin-Wagner theorem

Last time, we showed that Spontaneous breaking of continues symmetry $\Rightarrow$ gapless Nambu-Goldstone modes

But the analysis is based on mean-field theory, we want to know if this statement is still true even after considering the quantum fluctuations. Actually we will show that there is NO SSB (continuous symmetry) in $d \leq 2$, since fluctuation destroy the long-range order.

The effective Hamiltonian of NG modes for $O(2)$-model is

$$H_{eff} = \frac{k}{2\pi} \int d^d r \, (\vec{\nabla} \theta(\vec{r}))^2$$

From the microscopic lattice model

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

Since $Z := \mathrm{Tr}\, e^{-\frac{H}{k_B T}}$, for small temperature, $|\theta_i - \theta_j| \ll 1$

$$H \simeq -J \sum_{\langle ij \rangle} \left(1 - \frac{1}{2}(\theta_i - \theta_j)^2\right)$$

$$H \simeq \frac{J}{2} \sum_i \frac{z_i}{2} a^2 \left(\frac{\theta_i - \theta_j}{a} \right)^2 + \text{const}$$

Here z_i means the number of connected sites. By taking continue limit, i.e. $a \to 0$, we obtain:

$$H \xrightarrow{a \to 0} \frac{Jz}{4 d a^{d-2}} \int d^d \vec{r} \, (\vec{\nabla} \theta)^2$$

Compare with the H_{eff},

$$\frac{K}{2\lambda} = \frac{Jz}{4 d a^{d-2}} \cdot \frac{1}{k_B T} \propto \frac{1}{T}$$

which means $K \to \infty$ as $T \to 0$ and $K \to 0$ as $T \to \infty$

$H_{eff}[\theta]$ is a Gaussian theory at a critical point, any potential of θ is forbidden since $O(2)$ symmetry $(\theta \mapsto \theta + \delta)$

We already knew that in Gaussian theory

$$\langle \theta(\vec{r}) \, \theta(\vec{0}) \rangle \propto \begin{cases} \frac{1}{|\vec{r}|^{d-2}} & , d \neq 2 \\ \log |\vec{r}| & , d = 2 \end{cases}$$

(Just by solving $\nabla^2 G = -\delta$)

Let's regularize the correlation function by $\langle \theta(L) \theta(0) \rangle \sim 0$

Bondary condition.

here L is the size of the system.

$$\Rightarrow \langle \theta(\vec{r}) \theta(\vec{0}) \rangle \propto \begin{cases} \frac{1}{|\vec{r}|^{d-2}} - \frac{1}{L^{d-2}} & , d \neq 2 \\ \log \frac{|\vec{r}|}{L} & , d = 2 \end{cases}$$

$$\Rightarrow \langle \theta^2 \rangle \sim \langle \theta(a) \theta(0) \rangle \propto \begin{cases} \frac{1}{a^{d-2}} - \frac{1}{L^{d-2}} & , d \neq 2 \\ \log \frac{a}{L} & , d = 2 \end{cases}$$

So we both have large distance cut off, L, and short distance cut off, a. Take $L \to \infty$ limit,

$$\langle \theta^2 \rangle \xrightarrow[L\to\infty]{} \begin{cases} \infty & , d \le 2 \\ \text{finit} & . d > 2 \end{cases}$$

Therefore, $d \le 2$ system has long-distance (IR) divergence

Now back to the order parameter $\vec{\phi} = (\phi_1, \phi_2) = (\cos\theta, \sin\theta)$

e.g. $\langle \phi_1 \rangle = \langle \cos\theta \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \langle \theta^{2n} \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot (2n-1)!! \langle \theta^2 \rangle^n$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} \cdot (-\langle \theta^2 \rangle)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\langle \theta^2 \rangle}{2}\right)^n = \exp\left(-\frac{\langle \theta^2 \rangle}{2}\right)$$

It means $\langle \phi_1 \rangle \xrightarrow{L\to\infty} 0$ in $d \le 2$. Quantum fluctuation makes $\langle \vec{\phi} \rangle$ at the tip of Mexico hat! ($\langle \vec{\phi} \rangle = 0$)

Hence, there is no spontaneous breaking of $O(2)$ symmetry in $d \le 2$. This arguement can be generalized to any continuous symmetry.

Nevertheless, the $O(2)$ theory in $d=2$ exhibits a unique physics. Recall that the normalization factor:

$$\langle \theta(\vec{r}) \theta(0) \rangle \sim \frac{1}{2K} \log |\vec{r}|$$

Consider the 2-pt correlation function.

$$\langle \vec{\phi}(\vec{r}) \cdot \vec{\phi}(0) \rangle = \langle \cos(\theta(\vec{r}) - \theta(\vec{0})) \rangle$$

By the same method we used for computing $\langle \phi_1 \rangle$

$$\langle \vec{\phi}(\vec{r}) \cdot \vec{\phi}(0) \rangle = \exp\left[-\frac{1}{2} \langle (\theta(\vec{r}) - \theta(\vec{0}))^2 \rangle\right]$$

$$= \exp\left(\langle \theta(\vec{r}) \theta(0) \rangle - \langle \theta(\vec{0})^2 \rangle\right)$$

$$\sim \exp\left(-\frac{1}{2K} \log |\vec{r}| + \frac{1}{2K} \log a\right)$$

$$\sim \left(\frac{a}{|\vec{r}|}\right)^{\frac{1}{2K}}$$

In the second equality, we used translation invariance. $\langle \theta(\vec{r})^2 \rangle = \langle \theta(0) \rangle^2$. From the 2-pt correlation function, we know the critical exponent is $\frac{1}{2K}$, and we already know K is a parameter varying respect to T. However it's a highly non-trivial result, since any example we have met until now has the fixed critical exponent. This novelty phenomenon called quasi-long-range order.

For a generic Gaussian fixed point, the coupling constant can be absorbed by redefinition of the field operator. However, θ is an angle! $\theta \sim \theta + 2\pi$, so we can not rescale θ arbitrarily. So $\frac{1}{2K}$ is indeed observable. So in this case, we should expect a line of fixed points. not only a single point. The parameter K is exactly marginal named as Luttinger parameter of the Tomonaga-Luttinger liquid.

For $O(N>2)$, there are multiple gapless modes interact with each other. There are No quasi-long-range order and it's gapped.

§12 Berezinskii-Kosterlitz-Thousless transition

We know that the 2D XY model on a lattice becomes completely disordered at sufficiently high temperature, which means the following result seems wrong:

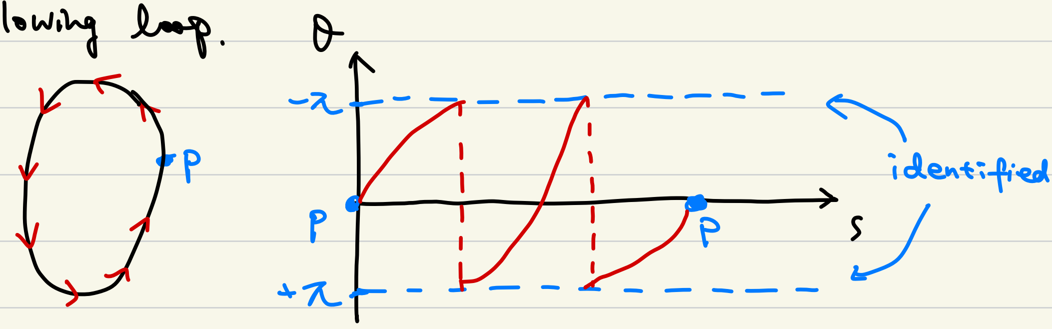
$$\langle \vec{\phi}(\vec{r}) \cdot \vec{\phi}(\vec{0}) \rangle \propto \frac{1}{r^{\frac{1}{2K}}}$$

So we muct missed something in our field theory analysis. Indeed, we lost a topological term in the previous analysis, and it's essential for phase transition despite the absense of SSB. So this is a novelty phase transition, not just drived by SSB. This is the first example of topological phase transition.

The key point is that the $\theta(\vec{r})$ is periodic in $[0, 2\pi)$

$$\theta + 2\pi \sim \theta \in \mathbb{R}/2\pi\mathbb{Z}$$

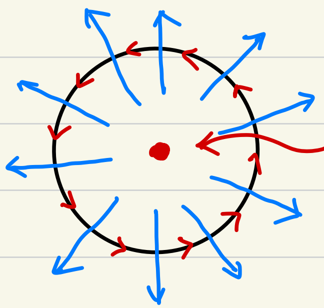
Which means it can "wind". For example consider the following loop.



Here we illustrate a possible configuration with non-trivial topology. obviously, it has winding number $W=2$. In general, the winding number can be computed by the following formulae:

$$\oint_C (\vec{\nabla}\theta)\cdot d\vec{r} = 2\pi W \quad , \quad W \in \pi_1(S^1) = \mathbb{Z}$$

e.g. $W=1$



"Votex" configuration.

$\theta(0)$ is ill-define, called topological defect

$\theta_i - \theta_j \ll 1$ is not true

However, in the previous lecture, we used the Tylor expansion of $\cos(\theta_i - \theta_j)$. This is not valid when the Votex singularity exists. So the contribution of Vortex can not be captured by Gaussian Theory.

The energy of a single Vortex can be computed by

$$E = \frac{K}{2\pi}\int d^2\vec{r}\,(\vec{\nabla}\theta)^2 = \frac{K}{2\pi}\int_0^L 2\pi r\,dr\cdot\frac{1}{r^2} = K\log\frac{L}{a}$$

And the entropy is given by

$$S = \log\left(\frac{L}{a}\right)^2 = 2\log\frac{L}{a}$$

\# of choices of single vortex configuration

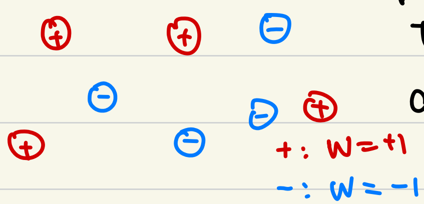
Then the free energy is

$$F = E - S = (k-2)\log\frac{L}{a} \xrightarrow{L\to\infty} \begin{cases} +\infty & k>2 \\ -\infty & k<2 \end{cases}$$

This means a vortex becomes stable at high temperature region (Recall that $k \sim \frac{1}{T}$) and unstable at low temperature region. So in the low temperature region, we can believe the result from Gaussian theory. Then we just see the quasi-long-range order

$$\langle \vec{\phi}(\vec{r}) \cdot \vec{\phi}(\vec{0}) \rangle \propto \frac{1}{|\vec{r}|^{\frac{1}{2k}}}, \quad T < T_c$$

But vortices are stable at $T > T_c$, In this case, one will encouter the proliferation and screening of vortices.



Then the short-range order would appear:

$$\langle \vec{\phi}(\vec{r}) \vec{\phi}(\vec{0}) \rangle \propto e^{-\frac{|\vec{r}|}{\xi}}$$

This mechanism is similar to Coulomb gas, the effective electric field is

Gauss law in 2D

$$\vec{E} = \begin{pmatrix} \partial_y \theta \\ -\partial_x \theta \end{pmatrix} \Rightarrow \oint_C \vec{E} \cdot \vec{n}\, ds = 2\pi W$$

Away from vortices, Gaussian theory works. It is described by the poisson equation

$$\nabla^2 \theta = 0$$

From the E-L eq. of $\mathcal{H}[\theta] \sim (\vec{\nabla}\theta)^2$. Obviously, this equation is just analog of $\vec{\nabla} \times \vec{E} = 0$. In conclusion, at high temperature region, we obtain the phase as Coulomb gas of vortices. This phase transition becomes from the topology of Vortex. called as BKT transition, a baby version of topological phase transition.

Now, we use anotherway, perturbative RG, to describe the BKT transition quantitatively. First, we write down the effective Hamiltonian including vortices

$$H_{eff} = \frac{K}{2\pi} \int d^2\vec{r} \, (\vec{\nabla}\theta)^2 - y \int d^2\vec{r} \left(V(\vec{r}) + V^{\dagger}(\vec{r}) \right)$$

y: effective weight of creating a $W = \pm 1$ vortex (fugacity)

$V(\vec{r})$: creation operator of a $W = +1$ Vortex at $\vec{r}$

$V^{\dagger}(\vec{r})$: ~ $W = -1$ ~

More precisely, $V(\vec{r})$ is defined as boudary condition of path integral.

$$\langle V(\vec{r}) \cdots \rangle = \int_{\substack{\text{Configuration of } \theta(\vec{r}) \\ \text{with } \int_{C_{\vec{r}}} d\vec{r} \cdot (\nabla\theta) = 2\pi}} D\theta(\vec{r}) \, e^{-H[\theta]} \cdots$$

The derivitation of H_{eff} is left in literature.

The Leading order approximation of perturbative RG equation is

at Gaussian fixed pt

$$\frac{dy}{dl} = (\overset{=2}{d} - x_y)\, y + \cdots , \quad \langle V(\vec{r}_1) V^\dagger(\vec{r}_2) \rangle_0 \propto \frac{1}{|\vec{r}_1 - \vec{r}_2|^{2x_y}}$$

From the definition of $\langle V(\vec{r}_1) V^\dagger(\vec{r}_2) \cdots \rangle$, we have

$$\langle V(\vec{r}_1) V^\dagger(\vec{r}_2) \rangle = \int D\theta(\vec{r})\, e^{-H_{eff}} \sim e^{-E(\vec{r}_1, \vec{r}_2)}$$

$E(\vec{r}_1, \vec{r}_2)$ is the energy of a $W=+1$ Vortex at $\vec{r}_1$ and $W=-1$ vortex at $\vec{r}_2$. Since the behaviour of Vortices is similar to Coulomb gas, $E(\vec{r}_1, \vec{r}_2)$ can be solved as

$$E(\vec{r}_1, \vec{r}_2) = 2K \log \frac{|\vec{r}_1 - \vec{r}_2|}{a}$$

$\Rightarrow \; x_y = K$, therefore.

$$\frac{dy}{dl} = (2-K)\, y + \cdots$$

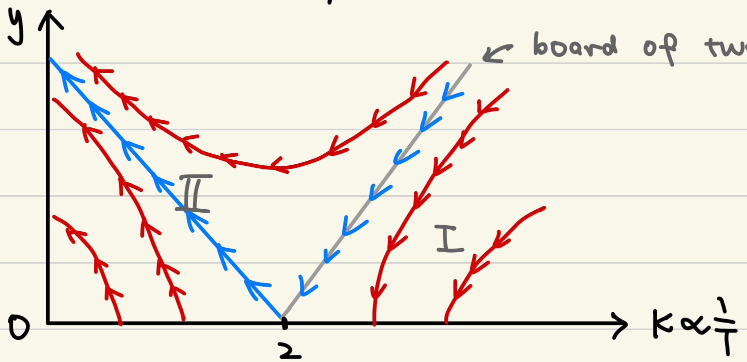
$x_K = 2$ is given by the original Gaussian theory. so the leading order of β_K vanishes. We need to tackle non-linear contributions which is tedious

$$\frac{dK}{dl} = -Ay^2 - 2ByK - CK^2$$

However, CK^2 should vanish since $y=0$ is a solution of the perturbative RG equation as the Gaussian fixed pt. And $2ByK$ also should vanish by the analysis of symmetry $Z(y) = Z(-y) \Rightarrow y$-odd term should vanish.

Only one term, Ay^2, left. The explicit expression of A is not necessary in our analysis. The trajector can be written down as:

$$Ay^2 - (2-K)^2 = \text{const.}$$



phase I : $K|_{l\to\infty} > 0$, $y|_{l\to\infty} = 0$

quasi-long-range.

$K_c = 2$ is universal.

phase II: $K|_{l\to\infty} = 0$, $y|_{l\to\infty} = \infty$

Short-range (disordered)

short-range : $G(r) \sim e^{-r/\xi}$, $G(\infty) = 0$

long-range : $G(\infty) = \text{Const} \neq 0$

quasi-long-range: $G(\infty) = 0$ but $G(r) \sim r^{-\eta(T)}$

終わりです