Morse Theory.

Global Geometry Properties of Manifold

$$\Updownarrow$$

C^∞ Function on the Manifold

The Simplest Example

2-dimension, Compact, Smooth, Orientable Manifold.

Classify to gT^2 and KP^2

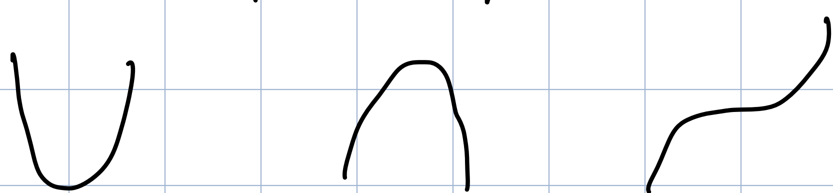
Consider gT^2, $f \in C^\infty(M)$, Calculate its Critical point p. which satisfy the eq:

$$\boxed{df|_p = 0} \iff \frac{\partial f}{\partial x_1}\Big|_p = \frac{\partial f}{\partial x_2}\Big|_p = 0$$

Hessian Matrix:

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \qquad \boxed{\det H \neq 0}$$

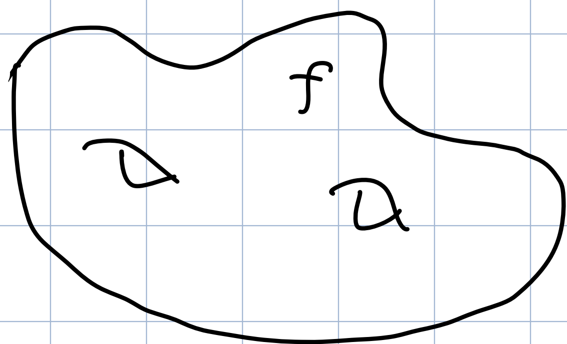
We consider f whose critical points are non-degenerate. $\det H_p \neq 0 \quad \forall p$.



How many non-degenerate critical point of f

$$\left.\begin{array}{l}\text{Maximum point} \geq 1\\ \text{Minimum point} \geq 1\\ \text{Saddle point} \geq 2g\end{array}\right\}$$ The simplest Morse inequation.

link a Manifold to a Quantum System.



$\Longrightarrow \hat{H}, \{|\psi\rangle\}$

Wyle operator

Quantization: Observeable $f(x,p) \to \hat{f}(\hat{x},\hat{p})$

but $[\hat{x},\hat{p}] = i\hbar \neq 0$

classical observable Algebra is commutive

But Quantum observable is not.

$$f = \sum_m f_m(x) p^m.$$

$$f^{+} v = \sum_m f_m(x) \left(-i\hbar \frac{\partial}{\partial x}\right)^m v(x).$$

$$= \sum_m \left(-i\hbar \frac{\partial}{\partial x}\right)^m f_m(y) v(x) \Big|_{y=x}$$

$$f^{-}(U) = \sum_m \left(-i\hbar \frac{\partial}{\partial x}\right)^m [f_m(x)\, U(x)]$$

$$= \sum_m \left(-i\hbar \frac{\partial}{\partial x}\right)^m f_m(x)\, U(x)\Big|_{y=0}$$

$$\Rightarrow \hat{f}_{wyle}\, U \equiv \sum_m \left(-i\hbar \frac{\partial}{\partial x}\right)^m f_m\left(\frac{x+y}{2}\right) U(x)\Big|_{y=x}$$

Semi-classical limit

$$\underline{\{\hat{f}, \hat{g}\}_q = \frac{1}{i\hbar}[\hat{f}, \hat{g}]} \Rightarrow \{f, g\} \to \{\hat{f} \cdot \hat{g}\}_q$$

if $\hbar \to 0$ $\Rightarrow$ $\hat{f}\hat{g} = \widehat{fg} + O(\hbar)$

$$\{\hat{f}, \hat{g}\}_q = \widehat{\{f, g\}} + O(\hbar)$$

proof

$f(x,p)$, $g(x,p) \propto p^m$ $\quad p = -i\hbar\partial_x$, $F(x,p) = a(x)\, p^m$

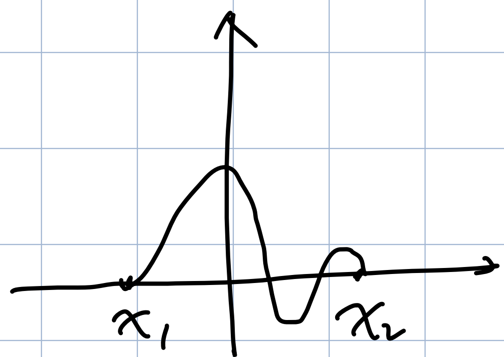
$$\hat{p}\hat{F}U = \left(-i\hbar \frac{\partial}{\partial x}\right)\left(\left(-i\hbar\frac{\partial}{\partial x}\right)^m a\left(\frac{x+y}{2}\right) U(x)\Big|_{y=x}\right)$$

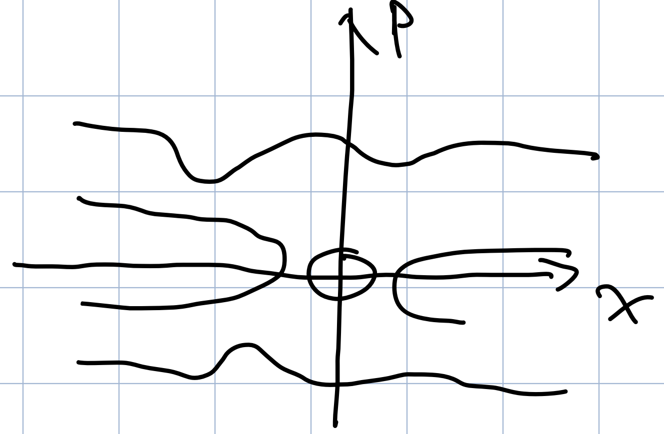
$$= \left(-i\hbar \frac{\partial}{\partial x}\right)^{m+1} a\left(\frac{x+y}{2}\right) U(x)\Big|_{y=x}$$

$$+\left(-i\hbar\frac{\partial}{\partial y}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)^m a\left(\frac{x+y}{2}\right) U(x)\Big|_{y=x}$$

$$= \widehat{pF} - i\hbar \cdot \frac{1}{2}\widehat{\frac{\partial F}{\partial x}}$$

$$\hat{F}\hat{p} = \widehat{pF} + \frac{i\hbar}{2}\widehat{\frac{\partial F}{\partial x}}$$





$H(x,p)=E$

QM 核心是解 time-independent Schrödinger eq.

$$-\frac{\hbar^2}{2}\psi'' + V(x)\psi = E\psi \Longrightarrow \text{解空间 } \mathcal{L}.$$

考虑 $E>0$, 在 $(-\infty, x_1)\cup(x_2,+\infty)$ 上 Schrödinger Eq.

$$-\frac{\hbar^2}{2}\psi'' = E\psi \Longrightarrow \text{解空间 } \mathcal{L}_0 = \mathrm{span}\{\sin,\cos\}$$

$\mathcal{L}$ 与 $\mathcal{L}_0$ 都是二维线性空间.

单值化算子:

$B_\pm : L \to L_0$. $\quad u(x)\in L$, $u_0(x)\in L_0$

$B_- u(x) = u_0(x)$. 且 $u = u_0|_{x<x_1}$

$B_+ u(x) = u_0(x)$ 且 $u = u_0|_{x>x_2}$

$B_\pm \in \mathcal{L}(L, L_0)$ 且由 ode 存在唯一定理 $B_\pm$ 是同构

$$M = B_+ B_-^{-1} : L_0 \to L_0$$

单值化算子.

M 其实就是散射矩阵

取 L_0 基底 $e_1 = \sin(kx)$, $e_2 = \cos(kx)$, $k^2 \equiv \frac{2E}{\hbar^2}$

$\eta, \xi \in L_0$. define $[\xi, \eta] \equiv \xi_1 \eta_2 - \xi_2 \eta_2$

即 $\left.\begin{aligned} \xi &= \xi_1 e_1 + \xi_2 e_2 \\ \eta &= \eta_1 e_1 + \eta_2 e_2 \end{aligned}\right\} \Rightarrow \xi \wedge \eta \equiv [\xi, \eta]$

<u>M保 $\xi\wedge\eta$</u>: $\xi \wedge \eta = M(\xi) \wedge M(\xi)$

即 M, $\wedge$ 可交换. 为把此过程. 可引入 L中 辛积

$\{\psi, \varphi\} \equiv \psi' \varphi - \psi \varphi'$. 即 朗斯基行列式

这与 $\xi\wedge\eta$ 一样其实不 depend on x.

$$\frac{d}{dx}\{\psi, \varphi\} = \psi'' \varphi + \psi' \varphi' - \psi' \varphi' - \psi \varphi'' = \psi'' \varphi - \psi \varphi''$$

$$\frac{\hbar}{2} \psi'' + V(x) \psi = E \psi, \quad \frac{\hbar}{2} \varphi'' + V(x) \varphi = E \varphi$$

$$\Rightarrow \frac{\hbar}{2} \psi'' \varphi + V(x) \psi \varphi = E \psi \varphi, \quad \frac{\hbar}{2} \varphi'' \psi + V(x) \psi \varphi = E \psi \varphi$$

$$\Rightarrow \psi'' \varphi - \varphi'' \psi = 0$$

$\psi, \varphi \in L$, $x < x_1$ $\begin{cases} \psi = \xi_1 e_1 + \xi_2 e_2 \\ \varphi = \eta_1 e_1 + \eta_2 e_2 \end{cases}$

$$\{\psi, \varphi\} = \{\psi, \varphi\}_{x < x_1} = K(-\xi_1 \eta_2 + \xi_2 \eta_1) = -k[B_-\psi, B_-\varphi]$$

$$\{\psi, \varphi\} = -K\,[B_-\psi, B_-\varphi]$$

$\{\psi, \varphi\} = -K[B_+\psi, B_+\varphi]$

$$[M\xi, M\eta] = [B_+B_-^{-1}(\xi), B_+B_-^{-1}(\eta)]$$
$$= -\frac{1}{K}\{B_-^{-1}(\xi), B_-^{-1}(\eta)\} = [\xi, \eta]$$

Corollary $\underline{\det M = 1}$. 纵于作 $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\mathcal{L}, \mathcal{L}_0$ 都是实函数空间. 但更多信的作用是在.
$\mathcal{L}, \mathcal{L}_0$ 复化. 因为量子力学散射问题更多是行波 $e^{\pm ikx}$
$e^{\pm ikx} \notin \mathcal{L}/\mathcal{L}_0$ 而是它们复化空间之元素.

$$\mathcal{L}_0 \xrightarrow{\text{复化}} {}^{\mathbb{C}}\mathcal{L}_0 \qquad \{f_1 = e_1 + ie_2,\ f_2 = e_1 - ie_2\}$$

$\langle \xi, \eta \rangle = \frac{1}{2i}[\xi, \bar{\eta}]$. 在 ${}^{\mathbb{C}}\mathcal{L}_0$ 下 ξ, η 系数 $\in \mathbb{C}$
内积 但不正定. 故只成了 Hermite form.

$$\langle f_i, f_j \rangle = \eta_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$A : {}^{\mathbb{C}}L_0 \to {}^{\mathbb{C}}L_0$. 这时 A 都非退化 (可逆)

A 保 ${}^{\mathbb{C}}L_0$ 上 $[\,,\,]$ 运算. 构成一个群.

$$Sp(1, \mathbb{C}) \cong SL(2, \mathbb{C})$$

保 ${}^{\mathbb{C}}L_0$ 上 $\langle\,,\,\rangle$ 运算的 构成群 $U(1,1)$

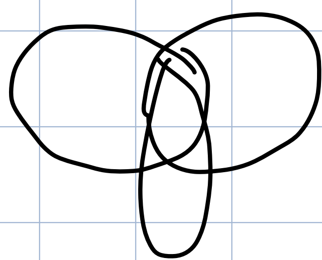
$A\bar{\xi} = \overline{A\xi}$ 则 $A \in GL(2,\mathbb{R})$

相容于 保 $\mathbb{C}_{\mathcal{L}}$ 上的共轭运算这一代数结构

$$Sp(1,\mathbb{C}) \cap U(1,1) \in GL(2,\mathbb{R})$$

$$Sp(1,\mathbb{C}) \cap GL(2,\mathbb{R}) \in U(1,1)$$

$$U(1,1) \cap GL(2,\mathbb{R}) \in Sp(1,1)$$



Proof is obviously.

$$Sp(1,\mathbb{C}) \cap U(1,1) \cap GL(2,\mathbb{R}) = SL(2,\mathbb{R}) \cong SU(1,1)$$

这 其实是 M 所构成的群！ 数学家们 M 为 $SU(1,1)$ 与 $SL(2,\mathbb{R})$ 同构！

所以 M 保 $\langle , \rangle$. 即 保 $I = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

即 $\boxed{M^{\dagger} I M = I}$

$$M = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}$$

$$M^{\dagger} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix}$$

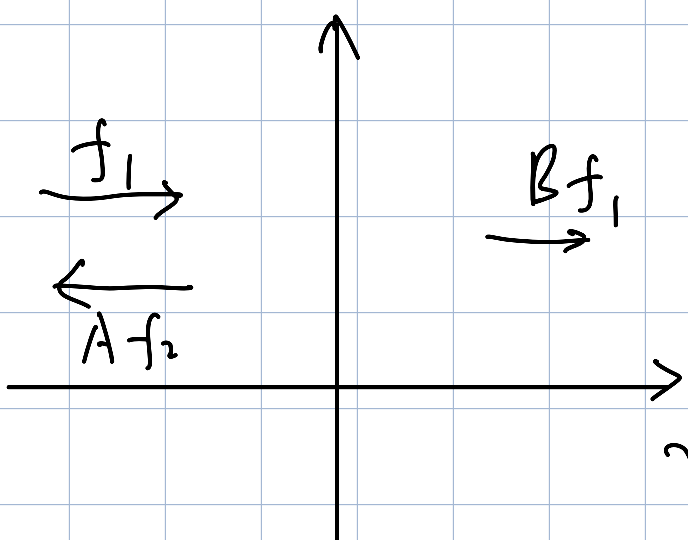
$$M^{\dagger} I M = I$$

$$\Rightarrow \begin{pmatrix} -\bar{\alpha} & \bar{\beta} \\ -\bar{\gamma} & \bar{\delta} \end{pmatrix} = \begin{pmatrix} -\delta & \gamma \\ -\beta & \alpha \end{pmatrix}$$

$$\delta = \bar{\alpha}, \quad \beta = \bar{\gamma}$$

$$\Rightarrow M = \begin{pmatrix} \alpha & \bar{\beta} \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$ 在 f_1, f_2 基下

由 $\det M = |\alpha|^2 - |\beta|^2 = 1 \Rightarrow \alpha \neq 0$.



此边界条件下 Schrödinger 方程有唯一解.

$$\psi = \begin{cases} f_1 + r f_2, & x < x_1 \\ r f_1, & x > x_2 \end{cases}$$

只用证 $\exists M$. s.t. $M\begin{pmatrix}1\\ r\end{pmatrix}=\begin{pmatrix}\tau\\ 0\end{pmatrix}$

$\tau, r \in \mathbb{C}$

$$\Rightarrow \begin{cases} \alpha + r\bar{\beta} = \tau \\ \beta + r\bar{\alpha} = 0 \end{cases} \Rightarrow \begin{cases} r = -\frac{\beta}{\alpha} \\ \tau = \alpha - \frac{|\beta|^2}{\alpha} = \frac{1}{\alpha} \end{cases}$$

$$T \equiv |\tau|^2, \quad R \equiv |r|^2$$

$$\Rightarrow R + T = \frac{|\beta|^2}{|\alpha|^2} + \frac{1}{|\alpha|^2} = 1$$

$T = \frac{1}{|\alpha|^2} \neq 0$、but $R = 0$ 是可以的

谐振

$R \neq 0$. $E > V_{max}$ 也会反射

这只是数学上形式化的说法，真正计算还得看物理书

量子谐振子

$$H(x,p)=\frac{1}{2}p^2+\frac{1}{2}\omega^2x^2 \Rightarrow \hat{H}=-\frac{\hbar^2}{2}\frac{d^2}{dx^2}+\frac{\omega^2x^2}{2}$$

$$\hat{a}=\omega x+\hbar\frac{d}{dx}, \quad \hat{a}^{+}=\omega x-\hbar\frac{d}{dx}$$

$$\Rightarrow \hat{H}=(\hat{a}\hat{a}^{+}-\hbar\omega)\cdot\frac{1}{2}$$

$$[\hat{a},\hat{a}^{+}]=2\hbar\omega, \quad [\hat{H},\hat{a}]=-\hbar\omega\hat{a}, \quad [\hat{H},\hat{a}^{+}]=\hbar\omega\hat{a}^{+}$$

谱性质: $\hat{H}=\frac{1}{2}\hbar\omega+\frac{1}{2}\hat{a}^{+}\hat{a}$, $E\in\mathbb{C}$.

$$E\psi=\frac{1}{2}\hat{a}^{+}\hat{a}\psi+\frac{1}{2}\hbar\omega\psi$$

$$\Rightarrow E(\psi,\psi)=\frac{1}{2}\hbar\omega(\psi,\psi)+\frac{1}{2}(\hat{a}^{+}\hat{a}\psi,\psi)$$

$$=\frac{1}{2}\hbar\omega(\psi,\psi)+\frac{1}{2}(\hat{a}\psi,\hat{a}\psi)$$

$$\Rightarrow E=\frac{1}{2}\hbar\omega+\frac{1}{2}\frac{\|\hat{a}\psi\|^2}{\|\psi\|^2}\geq\frac{1}{2}\hbar\omega$$

下证可以取到 $\frac{1}{2}\hbar\omega$，只用证明 $\hat{a}\psi_0=0$ 有解即可.

$$\hat{a}\psi_0=0\Rightarrow \hbar\psi_0'+\omega x\psi_0=0 \Rightarrow \psi_0=Ce^{-\frac{\omega x^2}{2\hbar}}$$

利用物理书上方法可进一步 proof $E=(n+\frac{1}{2})\hbar\omega$, 且 $\psi_n\sim(\hat{a}^{+})^n\psi_0$

本质上是在找 sl_2 的表示

准经典极限

$$\psi_0\sim e^{-\frac{\omega}{2}\left(\frac{x}{\sqrt{\hbar}}\right)^2}$$

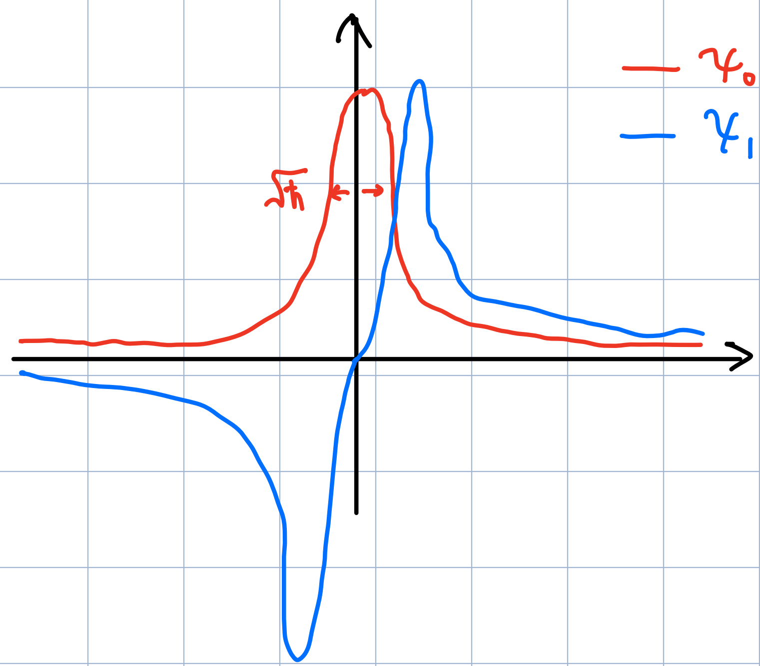
$$\hat{a}^{+}\sim\sqrt{\hbar}\left(\omega\frac{x}{\sqrt{\hbar}}-\frac{d}{d(x/\sqrt{\hbar})}\right)$$

取 $\xi = \frac{x}{\sqrt{\hbar}}$

$$\Rightarrow \psi_0 \sim e^{-\frac{\omega}{2}\xi^2}, \quad \hat{a}^\dagger \sim \sqrt{\hbar}\left(\omega\xi - \frac{d}{d\xi}\right)$$

$$\psi_m = P_m(\xi)\, e^{-\frac{\omega\xi^2}{2}} = \hbar^{-\frac{1}{4}} f_m\left(\frac{x}{\sqrt{\hbar}}\right), \quad \|f_m\|^2 := 1$$

且 $x \to \infty$, $f_m \to e^{-x^2} \to 0$



高维情形.

$$H(x,p) = \frac{1}{2}|p|^2 + \frac{1}{2}(x, \Omega^2 x)$$

$$\Omega^2(x) \in SO^+(3) \xrightarrow[\text{主轴}]{\text{取惯量}} \Omega^2 \Rightarrow \Lambda^2 = (\omega_1^2, \dots, \omega_n^2)$$

$$\hat{H} = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}(x, \Omega^2 x), \quad \Delta := \nabla^2$$

找到正交基底：

$$\hat{H} = -\frac{\hbar^2}{2}\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} + \frac{1}{2}\sum_{j=1}^{n} \omega_j^2 y_j^2$$

$$= \sum_{j=1}^{n} \hat{H}_j^{1d}$$

$$\Rightarrow E = \hbar \sum_{j=1}^{N} (m_j + \tfrac{1}{2}) \omega_j$$

$$\psi_m = \prod_{j=1}^{N} (a_j^+)^{m_j} \, e^{-\frac{1}{2\hbar} \sum_{j=1}^{N} \omega_j^2 y_j^2}$$

ψ_0^{nd}

$$= C_m P_m\left(\frac{x}{\sqrt{\hbar}}\right) e^{-\frac{1}{2}\left(\frac{x}{\sqrt{\hbar}}, \Omega \frac{x}{\sqrt{\hbar}}\right)}$$

$$= \hbar^{-n/4} f_m\left(\frac{x}{\sqrt{\hbar}}\right) \qquad \|f_m\|^2 = 1$$

$x \in \mathbb{R}^n$, $\hat{H} = -\frac{\hbar^2}{2} \Delta + V(x)$

在极值点处 Taylor 展开可看做谐振子.

$$\hat{H}_0 = -\frac{\hbar^2}{2} \Delta + V(x_0) + \frac{1}{2} \langle x - x_0, V''(x - x_0) \rangle$$

设 V'' Hessian 矩阵非退化.

$$E_m = V(x_0) + \sum_{j=1}^{n} \hbar \omega_j (m_j + \tfrac{1}{2}), \quad \omega_j^2 \text{ 为 } V'' \text{ 本征值}$$

$$V''_{ij} = \left.\frac{\partial V}{\partial x_i \partial x_j}\right|_{x = x_0}.$$

$$\psi_m = \hbar^{-n/4} f_m\left(\frac{x - x_0}{\sqrt{\hbar}}\right), \quad \|f_m\|^2 = 1$$

引理: 设 $W(x) \in C^\infty(\mathbb{R}^n)$, 且 $|x| \to \infty$ 时 增速不超过多项式速度 $x \to x_0$ 时 $W(x) = O(|x - x_0|^s)$

则 $\|W(x) \psi_m\| = O(\hbar^{s/2})$

Proof: $\|W(x)\chi_m\|^2 = \hbar^{-n/2}\int_{\mathbb{R}^n} W(x)^2 f_m^2\left(\frac{x-x_0}{\sqrt{\hbar}}\right)dx$

鞍点近似 $= \hbar^{-n/2}\int_{|x-x_0|<\delta} W^2(x) f_m^2\left(\frac{x-x_0}{\sqrt{\hbar}}\right)dx + O(\hbar^{\infty})$

$\leq \hbar^{-n/2} C\int_{|x-x_0|<\delta} |x-x_0|^{2s} f_m^2\left(\frac{x-x_0}{\sqrt{\hbar}}\right)dx$

$\overset{\xi=\frac{x}{\sqrt{\hbar}}}{\sim} \hbar^{-n/2} C\int_{\mathbb{R}^n} d\xi\, \xi^{2s}\hbar^{s} f_m^2(\xi)\cdot\hbar^{n/2}$

$\sim \hbar^{s}\int_{\mathbb{R}^n} d\xi\, \underbrace{\xi^{2s} f_m^2(\xi)}_{O(1)}\, C$

$\sim O(\hbar^{s})$

局部振子近似定理：令 x_0 为 $V(x)$ 的非退化极小值点，且设 $V(x)\in C^{\infty}(\mathbb{R}^n)$ 在 $|x|\to\infty$ 时增长不大于多项式，ψ_m 是 $\hat{H}_0$ 本征值。则 $(\hat{H}-\hat{H}_0)\psi_m = f$ 且 $\|f\| = O(\hbar^{3/2})$

Proof: $V(x) = V(x_0)+\frac{1}{2}(x-x_0, V''(x-x_0))+W$

$W(x) = O(|x-x_0|^3)$

$\Rightarrow \hat{H} = \hat{H}_0 + W(x)$

$\Rightarrow (\hat{H}-\hat{H}_0)\psi_m = W(x)\psi_m \Rightarrow \|W(x)\chi_m\| \sim O(\hbar^{3/2})$

上述论：若 $\hat{H}$ 是 ~~厄米~~ 的（自伴随）。则 $\forall m$，$\exists \lambda \in \sigma(\hat{H})$

$$|\lambda - E_m| = O(\hbar^{3/2})$$

From 泛函分析：$\hat{H}$ 自伴随 且 $\|(\hat{H}-E_m)^{-1}\| = \frac{1}{d(E_m)}$

其中 $d(E_m)$ 是 E_m 与 $\hat{H}$ 谱集之间距离

$$(\hat{H} - \psi_m)\psi_m = f \Rightarrow \psi_m = (\hat{H} - \psi_m)^{-1} f$$

$$1 = \|\psi_m\| \leq \|(\hat{H} - E_m)^{-1}\| \cdot \|f\|$$

$$\leq \frac{1}{d(E_m)} \|f\| \Rightarrow \underset{\uparrow\ |\lambda - E_m|}{d(E_m)} \leq \|f\| \sim O(\hbar^{3/2})$$

这说明 E_m 不是谱，但距近 $\hat{H}$ 的谱，被称为 $\hbar^{3/2}$ 伪谱

整体振子近似定理.

$\hat{H} = -\frac{\hbar^2}{2}\Delta + V(x)$，若 $V(x) \in C^\infty(\mathbb{R}^n)$

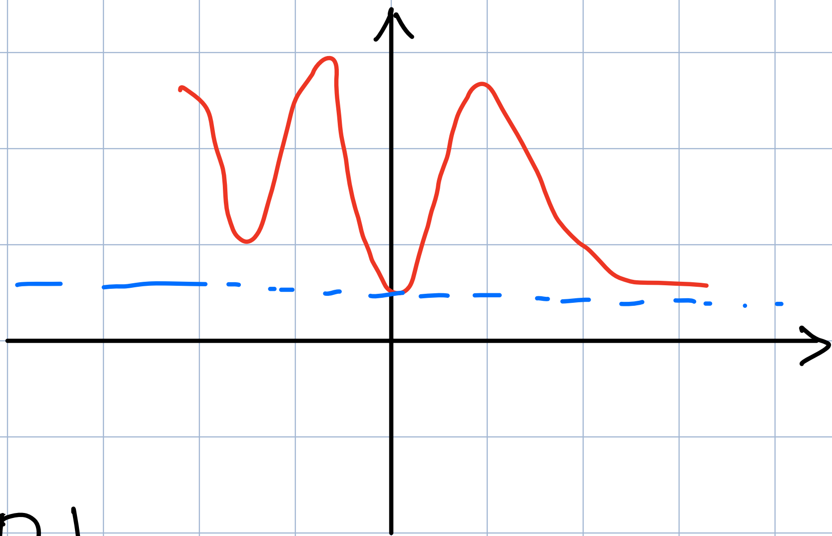
且有 N 个全局最小值点 $\{x^1, \cdots, x^N\}$ 且

都是非退化的极点. 那么每个 x^i 可以给出一

个伪谱 $E_m^{(j)} = V_0 + \sum_{k=1}^{n} \hbar(\omega_k^{(j)} + \frac{1}{2})$

若 $\exists$ 紧集 K 使得 $|V - V_0| \geq \delta > 0$ 在紧集外成

立. 即下图情况不可发生：



在 $|x| \to \infty$ 时

V 不能 $\to V_0$

则 $\forall M>0$，在 $\hbar$ 足够小时，至少存在 M 个 $\hat{H}$ 的特征值 $\{\lambda_s\}$ 有

$$\lambda_s = \mathcal{E}_0^s + O(\hbar^{3/2})$$

这里 λ_s 是 $\hat{H}$ 特征值考虑重数的升序排列.

$\mathcal{E}_0^s$ 在 $\hat{H}_0^{(j)}$ 特征值考虑重数的升序排列

前面局部的定理只说明了入存在性. 全局定理更强. 是从低到高能量的一一对应（至少M个）

Morse 理论

$f\in C^\infty(M)$. $\dim M:=n$.

p 称为 f 的临界点，若 $d_pf=0$，坐标形式即 $\nabla_p f=0$，称临界点 p 非退化若 $\partial_i\partial_j f$ 是非退化的二次型（无 0 特征值），称 $\partial_i\partial_j f$ 负特征值的个数称为 f 在 p 点的指数

Remark: 这些定义是与坐标无关的

Morse 引理：p 是 f 非退化临界点，则 $\exists$ p 邻域 U 上 chart $(y_1,\dots,y_n)$，使 $f(y)=f(p)+\sum_{j=1}^n \varepsilon_j y_j^2$

$\varepsilon_j=\pm1$. $\varepsilon_j=-1$ 个数是 f 在 p 点指数.

Proof：为简便取 $f(p)=0$. 用数学归纳法证 $k\le n$

$$f(y)=\sum_{j=1}^k \epsilon_j y_j^2+\sum_{i,j=k+1}^n Q_{ij}(y)y_iy_j \text{ 且}$$

Q_{ij} 非退化

① $k=0$：设 $x_j(p)=0$.

$$f(x)=\int_0^1\frac{d}{dt}f(tx)\,dt=\int_0^1\sum_{j=1}^n\frac{\partial f}{\partial x_j}(tx)\cdot x_j\,dt$$

$$=\sum_{j=1}^n x_j h_j(x),\quad h_j(x):=\int_0^1\frac{\partial f}{\partial x_j}(tx)\,dt$$

$h_j(p)=0$，构造 X 的剖 $\Rightarrow h_j(x)=\sum_{i=1}^{n} x_i Q_{ij}(x)$

$$\Rightarrow f(x)=\sum_{i,j} x_i x_j Q_{ij}(x)$$

$$\Rightarrow Q_{ij}(0)=\frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j} \Rightarrow Q_{ij} \text{ 非退化}$$

$\Rightarrow Q_{ij}(x)$ 在某邻域内退化

② $K\neq 0$ for $K+1$

$$f(y)=\sum_{j=1}^{K}\varepsilon_j y_j^2+\sum_{i,j} Q_{ij}(y) y_i y_j$$

$$\xrightarrow[\text{对角化 } Q_{ij}(0)]{y\to\tilde{y}} \sum_{j=1}^{K}\varepsilon_j \tilde{y}_j^2+Q_{K+1,K+1}(\tilde{y})\tilde{y}_{K+1}^2$$

$$+2\sum_{j=K+2}^{n} Q_{K+1,j}\tilde{y}_{K+1}\tilde{y}_j$$

$$+\sum_{i,j=K+2}^{n} Q_{i,j}(\tilde{y})\tilde{y}_i\tilde{y}_j$$

Q_{ij} 非退化 $\Rightarrow Q_{K+1,K+1}$ 在某邻域非零

$$Q_{K+1,K+1}=|Q_{K+1,K+1}|\underbrace{\operatorname{sgn}(Q_{K+1,K+1})}_{}$$

$$\to f(\tilde{y})=\sum_{j=1}^{K}\varepsilon_j\tilde{y}_j^2+\varepsilon_{K+1}\Big[\Big(\sqrt{|Q_{K+1,K+1}|}\,\tilde{y}_{K+1}\Big)^2$$

$$+2\tilde{y}_{K+1}\sqrt{|Q_{K+1,K+1}|}\sum_{j=K+2}^{n}\frac{Q_{K+1,j}\varepsilon_{K+1}}{\sqrt{|Q_{K+1,K+1}|}}\tilde{y}_j$$

$$+\Big(\sum^{n} \frac{Q_{K+1,j}\,\varepsilon_{K+1}}{\cdots}\Big)^2\Big]$$

$$\left.\left(\sum_{j=k+2}\frac{Q_{k+1,j}\,\varepsilon_{k+1}}{\sqrt{|Q_{k+1,k+1}|}}\tilde y_j\right)\right]$$

$$+\sum_{i,j=k+2}\tilde Q_{ij}(\tilde y)\,\tilde y_i\,\tilde y_j$$

$\tilde Q_{ij}$ 由 Q_{ij} 消去 $\varepsilon_{k+1}(\cdot)$ 得到.

令 $$z_{k+1}=\sqrt{|Q_{k+1,k+1}|}\,\tilde y_{k+1}+\sum_{j=k+2}^{n}\frac{Q_{k+1,j}\,\varepsilon_{k+1}\tilde y_j}{\sqrt{|Q_{k+1,k+1}|}}$$

$$z_{j\neq k}=\tilde y_j$$

则 $$f=\sum_{j=1}^{k+1}\varepsilon_j z_j^2+\sum_{i,j=k+2}^{n}\tilde Q_{ij}(z)\,z_i z_j$$

(1) 验证 $\tilde y\mapsto z$ 非奇异. $\Rightarrow$ Jacobi 非退化

(2) $\tilde Q_{ij}(z)$ 在原点处非退化.

(1) $$J(0)=\frac{\partial z_i}{\partial y_j}(0)=\begin{array}{c}\left[\begin{array}{ccccc}1&0&\cdots&\frac{\partial z_{k+1}}{\partial y_1}&0\\0&1&&\vdots&0\\\vdots&\vdots&&\vdots&\vdots\\&&&\sqrt{|Q_{k+1,k+1}|(0)}&\\&&&\vdots&\vdots\\0&0&&\vdots&1\end{array}\right]\leftarrow k+1\\ \uparrow\\ k+1\end{array}$$

利用 $Q_{ij}(0)$ 对角化, $Q_{k+1,k+1}(0)\neq 0\Rightarrow J(0)\neq 0$

(2) 证明 $|\tilde Q_{ij}(0)|\neq 0$

即 $$\frac{1}{2}\frac{\partial^2 f}{\partial x_i\partial x_j}(0)=\left(\begin{array}{c|c}\begin{matrix}\varepsilon_1&&0\\&\ddots&\\0&&\varepsilon_{k+1}\end{matrix}&\\\hline&\tilde Q_{ij}(0)\end{array}\right)$$

非退化 $\Rightarrow$ $\tilde{Q}_{ij}$ 非退化 $\square$

f 为 Morse 函数 iff 有有限个临界点且均非退化

设指数为 k 的临界点个数为 m_k

de Rham 上同调

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k \alpha\wedge d\beta$$

$$\alpha\wedge\beta = (-1)^{k+m}\beta\wedge\alpha \qquad \alpha\ k\text{-形式}.\ \beta\ m\text{-形式}$$

$$d(d\alpha) = 0$$

考虑 M 上 k-形式自由线性空间 $\Omega_0 = C^\infty$, $\Omega_{n+1} = 0$

$$0 \cdots \longrightarrow \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \longrightarrow \cdots 0$$

$$H^k(M) := \frac{\ker d_k}{\operatorname{Im} d_{k-1}} = Z_k / B_k$$

$$b_k = \dim H^k(M)$$

b_0: 连通分支个数. $H^1(M)$ 是 $\pi_1(M)$ 交换化

设 M 是光滑紧致 n 维可定向流形 且设 f 是 M 上 Morse 函数. 则

① $m_k \geq b_k$ 弱 Morse 不等式

② $\forall k$, $\sum_{j=0}^{k} (-1)^{k-j} m_j \geq \sum_{j=0}^{k} (-1)^{k-j} b_j$ 强 Morse 不等式

③ $\sum_{k=0}^{n} (-1)^k m_k = \sum_{k=0}^{n} (-1)^k b_k = \chi$ (Morse 指标定理)

设 V 是线性空间 $n := \dim V$.

则 $\Lambda^k := \sigma(\underbrace{V^* \otimes V^* \otimes \cdots V^*}_{k个})$ σ 为全反对称部分

$\dim \Lambda^k = C_n^k$ Λ^k 为 V 上 k-形式构成的空间

$\alpha \in \Lambda^k, \beta \in \Lambda^m$, $\alpha \wedge \beta \in \Lambda^{k+m}$

$$\alpha \wedge \beta (\xi_1, \cdots \xi_{k+m}) = \sum_{\substack{i_1 < \cdots < i_m \\ j_1 < \cdots < j_k}} \operatorname{sgn}(\sigma)\, \alpha(\xi_{i_1}, \cdots, \xi_{i_k}) \times \beta(\xi_{j_1}, \cdots, \xi_{j_m})$$

$\sigma = (i_1 \cdots i_k j_1 \cdots j_k)$ $\wedge$ 是结合的 所以后面用到的 $(e_1 \wedge \cdots e_n) \wedge (e^i \wedge \cdots \wedge e^k)$ 就是把括号去掉.

若 $\alpha_1, \cdots \alpha_k \in \Lambda^1$ 则:

$$\alpha_1 \wedge \cdots \wedge \alpha_k (\xi_1, \cdots \xi_k) = \det(\alpha_i(\xi_j))$$

取 V 中基 $e_1 \cdots e_n$, V^* 中对偶基 $e^1 \cdots e^n$

则 $e^{i_1} \wedge \cdots \wedge e^{i_k}$, $i_1 < \cdots < i_k$ 为 Λ^k 中基.

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$$

V 上若有内积 $\langle , \rangle$. 其诱导 $V \to V^* = \Lambda^1$ 上的同构 $\xi \mapsto G(\xi)$, $G(\xi)(\eta) = \langle \xi, \eta \rangle$

$\alpha, \beta \in \Lambda^1$ $\langle \alpha, \beta \rangle_1 = \langle G^{-1}(\alpha), G^{-1}(\beta) \rangle$ 即 Λ^1 上内积可由 V 上内积诱导

$g_{ij} := \langle e_i, e_j \rangle$ 则 $\langle e^i, e^j \rangle_1 = g^{ij} = (g^{-1})_{ij}$

下面定义 Λ^k 上内积，只用定义 $e^1 \wedge \cdots \wedge e^k$ 上内积即可.

$\alpha, \beta \in \Lambda^k$. $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$, $\beta = \beta_1 \wedge \cdots \wedge \beta_k$

则 $\langle \alpha, \beta \rangle_k := \det \langle \alpha_i, \beta_j \rangle_1$

这里 $\langle\, , \rangle_k$ 与从 $V^{\otimes k}$ 上定义内积再诱导到 $(V^*)^{\otimes k}$ 上
即利用 G^{-1}. 类似 Λ^1 内积定义. 然后限制到子空间 Λ^k 上相差
$k!$ 因子

Hodge 星算子

$V = \mathbb{R}^n$, 在上面取定向. 即给一个相容的体积外形式 Ω

① $\Omega \in \Lambda^n$ ② $\langle \Omega, \Omega \rangle_n = 1$

③ $\xi_1, \cdots \xi_n$ 定向在 V 中为正 $\Rightarrow \Omega(\xi_1, \cdots \xi_n) > 0$

$\dim \Lambda^k = \dim \Lambda^{n-k}$. $*$ 是 $\Lambda^k \to \Lambda^{n-k}$ 同构

$\alpha \in \Lambda^k$. $*\alpha \in \Lambda^{n-k}$, s.t. $\forall \beta \in \Lambda^{n-k}$

$$\langle *\alpha, \beta \rangle_{n-k} = \langle \alpha \wedge \beta, \Omega \rangle_n .$$

1) 设 $e_1 \cdots e_n$ 为标准正交基 且定向为正. 则

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \mathrm{sgn}(\sigma)\, e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$$

这里 $(j_1 \cdots j_{n-k}) := \{1, \cdots, n\} \setminus (i_1 \cdots i_k)$

$\sigma = (i_1 \cdots i_k, j_1 \cdots j_{n-k})$

proof: $\Omega = c\, e^1 \wedge \cdots \wedge e^n$ 而 $\langle \Omega, \Omega \rangle_n = 1$

$\Rightarrow c = \pm 1$, $e^1 \cdots e^n$ 正向 $\Rightarrow c = +1$

设 $\beta = e^{m_1} \wedge \cdots \wedge e^{m_{n-k}}$ 则

$$\langle *\alpha, \beta \rangle = \operatorname{sgn}\sigma \langle e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}, e^{m_1} \wedge \cdots \wedge e^{m_{n-k}} \rangle$$

$$= \operatorname{sgn}\sigma\, \delta^{j_1 \cdots j_{n-k}}_{m_1 \cdots m_{n-k}}$$

注意一直到现在的推导，我们使用 Λ^k 中基底 $e^{i_1} \wedge \cdots \wedge e^{i_k}$，$i_1 \cdots i_k$ 均是升序排列的。所以不会有 $\langle e^1 \wedge e^2, e^2 \wedge e^1 \rangle_2$ 这种问题

$$\langle \alpha \wedge \beta, \Omega \rangle = \langle e^{i_1} \wedge \cdots e^{i_k} \wedge e^{m_1} \wedge \cdots \wedge e^{m_{n-k}}, e^1 \wedge \cdots e^n \rangle$$

$$\propto \delta^{j_1 \cdots j_{n-k}}_{m_1 \cdots m_{n-k}}$$

比例系数已是置换 $\operatorname{sgn}\sigma$.

2). $\langle *\alpha, *\beta \rangle_{n-k} = \langle \alpha, \beta \rangle_k$.

由标准正交基在 $*$ 后仍是标准正交基 (1)，直接可得 (2)

3) $*(*\alpha) = (-1)^{k(n-k)} \alpha$. $\alpha \in \Lambda^k$

proof: $*(* e^{i_1} \wedge \cdots \wedge e^{i_k}) = *(e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}) \cdot \operatorname{sgn}\sigma$

$$= \operatorname{sgn}\sigma \operatorname{sgn}\rho\, e^{i_1} \wedge \cdots \wedge e^{i_k}$$

其中 $\rho = (j_1 \cdots j_{n-k}, i_1 \cdots i_k)$ 显然 $\operatorname{sgn}\rho = \operatorname{sgn}\sigma \cdot (-1)^{k(n-k)}$

4) $\langle *\alpha, \beta\rangle_{n-k} = (-1)^{k(n-k)} \langle \alpha, *\beta\rangle_k$

由 (2), (3) 直接推得

5) $\alpha \wedge *\beta = (\alpha, \beta)_k \Omega$

* 其实是在作正交补的操作

M 是 n 维定向无边紧致流形 上面有微分 k 形式 Ω^k.

取定 V 是 T_pM, $e_i = \frac{\partial}{\partial x^i}$ 内积是黎曼度量 g_{ij} 诱导

体积形式 Ω 给出定向，g_{ij} 诱导的 T_pM 上内积给

出了微分形式场 Ω^k 的内积 (,)

$$(\alpha, \beta) = \int_M \underbrace{\langle \alpha, \beta\rangle_p \Omega}_{\alpha\wedge *\beta}.$$

David Tong GR 有 physicist 版本

前面下标指示 是几阶形式 这里表示 位置

这里 $\langle , \rangle_p$ 表示在 T_pM^* 上由 T_pM 内积诱导的内积,

即 $\langle \alpha, \beta\rangle_1 = \langle G^{-1}(\alpha), G^{-1}(\beta)\rangle$ 类似定义. 然后积分给

出场的内积的恰当定义

由 g^{ij} 诱导

前提是 dx^i, $\frac{\partial}{\partial x^i}$ 的邦正交归一基

$d: \Omega^k \to \Omega^{k+1}$ $\quad d^*: \Omega^k \to \Omega^{k-1}$.

$$d^*\alpha := (-1)^{n+nk+1} * d * \alpha$$

d^* 其实与 d 在 (,) 下共轭 即 $(d\alpha, \beta) = (\alpha, d^*\beta)$

proof. $\alpha \in \Omega^k, \beta \in \Omega^{k+1}$

$$d(\alpha \wedge * \beta) = d\alpha \wedge * \beta + (-1)^k \alpha \wedge d * \beta$$

$$\int_M d(\alpha \wedge * \beta) = \int_{\partial M} \alpha \wedge * \beta = 0 \qquad (\partial M = \phi)$$

前面对p点处 TpM*上的公式可推广到M上Ω^k. 只用逐点内积即可，因为积分也是逐点积的

$$= \int_M d\alpha \wedge * \beta + (-1)^k \alpha \wedge d * \beta$$

$$= \int_M \Omega \langle d\alpha, \beta \rangle_p + (-1)^k \int_M \alpha \wedge d * \beta$$

$$= \int_M \Omega \langle d\alpha, \beta \rangle_p + (-1)^k \cdot (-1)^{k(n-k)} \int_M \alpha \wedge * * d * \beta$$

$$= \int_M \Omega \langle d\alpha, \beta \rangle_p + (-1)^k \cdot (-1)^{k(n-k)} \cdot (-1)^{n+n(k+1)+1} \int_M \alpha \wedge * d^* \beta$$

$$= \int_M \Omega \langle d\alpha, \beta \rangle_p + (-1) \int_M \Omega (\alpha, d^* \beta)$$

$$= 0 \qquad \square$$

d^* 也有时写为 $d^\dagger$

Laplace - Beltrami operator $\mathcal{D} : \Omega^k \to \Omega^k$

$$\mathcal{D} \alpha := (d d^* + d^* d) \alpha. \text{ (物理上记为 } \Delta \text{)}$$

$\mathcal{D}$ 射的性质

① $(\mathcal{D}\alpha, \beta) = (\alpha, \mathcal{D}\beta)$ $(\ ,\ )$ not $\langle\ ,\ \rangle_p$

② $(\mathcal{D}\alpha, \alpha) \geq 0$

③ $\mathcal{D}\alpha = 0 \iff \begin{matrix} d\alpha = 0 \\ d^*\alpha = 0 \end{matrix}$

④ $\ker \mathcal{D} \cong H^k(M)$，考虑 $\mathcal{D}: \Omega^k \to \Omega^k$

⑤ $[\mathcal{D}, *] = 0$

⑥ 如果假设是平坦 即 $g_{ij}(x) = \delta_{ij}$ 时)

$\mathcal{D}\, a(x)\, dx_{i_1} \cdots dx_{i_k} = -\Delta a\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

$$\Delta a = \sum_{k=1}^{n} \frac{\partial^2 a}{\partial x_k^2}$$

(多相同来)

Proof: ① $((dd^* + d^*d)\alpha, \beta) = (d^*\alpha, d^*\beta) + (d\alpha, d\beta)$

$= (\alpha, (dd^* + d^*d)\beta) = (\alpha, \mathcal{D}\beta)$

② $(\mathcal{D}\alpha, \alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha) \geq 0$

③ $\mathcal{D}\alpha = 0 \Rightarrow (\mathcal{D}\alpha, \alpha) = 0 \Rightarrow d\alpha = 0$ and $d^*\alpha = 0$

④ ② 自伴随. to $\Omega^k = \ker \mathcal{D} \oplus \operatorname{Im} \mathcal{D}$.

$\forall \omega$, $\omega = \omega_0 + \mathcal{D}\alpha$, $\omega_0 \in \ker \mathcal{D}$. $\alpha \in \Omega^k$

取 $\phi: Z^k \to \ker \mathcal{D}$ $\quad \omega \mapsto \omega_0$ 下面只用证 $\begin{cases} \mathrm{Im}\,\phi = \ker \mathcal{D} \\ B_k = \ker \phi \end{cases}$

$\forall\, \omega_0 \in \ker \mathcal{D}$ 由于分解唯一 故 $\phi(\omega_0) = \omega_0$ 这就证明了第一个

$d\omega = 0 \Rightarrow 0 = \underbrace{d\omega_0}_{0} + d\mathcal{D}\alpha = dd^*d\alpha$

0 因为 $\mathcal{D}\omega_0 = 0 \Rightarrow d\omega_0 = 0$

$dd^*d\alpha = 0 \Rightarrow 0 = (d\alpha, dd^*d\alpha) = (d^*d\alpha, d^*d\alpha)$

$\Rightarrow d^*d\alpha = 0$ 故 $\omega = \omega_0 + dd^*\alpha + d^*d\alpha = \omega_0 + d\underbrace{d^*\alpha}_{\beta}$

不能直接用 $dd^*d\alpha$ 因为在 d 之前 $d^*d\alpha$ 已经是 0

这里我们证明了 $\forall\, \omega \in Z^k$, $\omega = \omega_0 + d\beta$, $\omega_0 \in \ker \mathcal{D}$

若 $\omega \in \ker\phi \Rightarrow \omega_0 = 0 \Rightarrow \omega = d\beta \in B^k$

反之若 $\omega \in B^k$, $\omega = d\gamma = \omega_0 + d\beta$

$\Rightarrow \omega_0 = d(\gamma - \beta) := d\theta$ 而 $\mathcal{D}\omega_0 = 0 \Rightarrow d^*\omega_0 = d^*d\theta = 0$

$\Rightarrow (\theta, d^*d\theta) = 0 \Rightarrow (d\theta, d\theta) = 0 \Rightarrow d\theta = 0$

$\Rightarrow \omega_0 = 0 \Rightarrow \phi(\omega) = \omega_0 = 0 \Rightarrow \omega \in \ker\phi$

这就完成了证明. 但只是 physicist 的证明. 因为 $\mathcal{D}^+ = \mathcal{D}$

$\Rightarrow \Omega^k = \mathrm{Im}\,\mathcal{D} \oplus \ker \mathcal{D}$ 对有限维来说成立. $\mathcal{D}$ 无限维. 这一点是十分微妙的 需要泛函分析中的技巧完成

- $\mathcal{D}\omega = 0$ 称 ω 为调和形式

⑤ $\forall\, \alpha \in \Omega^k$

$$* \mathcal{D}\alpha = *(d^*d + dd^*)\alpha = **d*d(-1)^{n+n(k+1)+1}\alpha$$
$$+ *d*d*(-1)^{n+nk+1}\alpha$$

$$= \left[\underbrace{(-1)^{k(n-k)} \cdot (-1)^{n+n(k+1)+1}}_{(-1)^{k+1}} d * d + (-1)^{n+nk+1} * d * d * \right] \alpha$$

$$\mathcal{D} * \alpha = d^* d * \alpha + d d^* * \alpha = * d * d * \alpha \, (-1)^{n+n(n-k+1)+1}$$
$$+ \, d * d * * \alpha \, (-1)^{n+n(n-k)+1}$$
$$= \left[(-1)^{k+1} d * d + (-1)^{n+nk+1} * d * d * \right] \alpha$$

(b) $$\mathcal{D} \, a(x) \, dx_1 \wedge \cdots \wedge dx_k = d d^* a(x) \, dx_1 \wedge \cdots \wedge dx_k$$
$$+ \, d^* d \, a(x) \, dx_1 \wedge \cdots \wedge dx_k$$

$$d d^* a(x) \, dx_1 \wedge \cdots \wedge dx_k = d * d * (-1)^{n+nk+1} a(x) \, dx_1 \wedge \cdots \wedge dx_k$$
$$= d * d \, (-1)^{n+nk+1} a(x) \, dx_{k+1} \wedge \cdots \wedge dx_n$$
$$= d * (-1)^{n+nk+1} \sum_{i=1}^{k} \frac{\partial a}{\partial x^i} dx_i \wedge dx_{k+1} \wedge \cdots \wedge dx_n$$
$$= d \, (-1)^{n+nk+1} \sum_{j=1}^{k} \frac{\partial a}{\partial x_j} dx_1 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_k$$

$(-1)^{n+nk+j-1} = (-1)^{n-k+j-1+k(n-k)} \times \mathrm{sgn}(j, k+1, \ldots, n, 1, \ldots, \hat{j}, \ldots, k)$

$$= d \sum_{j=1}^{k} (-1)^j \frac{\partial a}{\partial x_j} dx_1 \wedge \cdots \widehat{dx_j} \cdots \wedge dx_k$$

$$= -\sum_{j=1}^{k} \frac{\partial^2 a}{\partial x_j^2} dx_1 \wedge \cdots \wedge dx_k$$

$$+ \sum_{j=1}^{k} \sum_{i=k+1}^{n} \frac{\partial^2 a}{\partial x_i \partial x_j} dx_1 \wedge \cdots \widehat{dx_j} \wedge \cdots dx_k \wedge dx_i$$

$(-1)^{j+k-1}$

$$d^* d\, a\, dx_1\wedge\cdots\wedge dx_k = d^* \sum_{i=k+1}^{n} \frac{\partial a}{\partial x_i}(-1)^k \cdot dx_1\wedge\cdots\wedge dx_k\wedge dx_i$$

$$= *d*(-1)^{n+n(k+1)+1}\sum_{i=k+1}^{n}\frac{\partial a}{\partial x_i}(-1)^k dx_1\wedge\cdots\wedge dx_k\wedge dx_i$$

$$= (-1)^{k+nk+1} * d\sum_{i=k+1}^{n}\frac{\partial a}{\partial x_i}(-1)^k dx_{k+1}\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots dx_k$$

$$\underbrace{(-1)^{i-k-1}}\ \ = * \operatorname{sgn}(1,\dots,k_i,k+1,\dots\hat{i}\dots n)$$

$$= (-1)^{nk} * \sum_{i=k+1}^{n}\frac{\partial^2 a}{\partial x_i^2} dx_{k+1}\wedge\cdots\wedge dx_n\,(-1)^{k+1}$$

$$+ (-1)^{nk} * \sum_{j=1}^{k}\sum_{i=k+1}^{n}\frac{\partial^2 a}{\partial x_j\partial x_i} dx_j\wedge dx_{k+1}\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n\,(-1)^i$$

$$= \underbrace{(-1)^{k(n-k)}(-1)^{nk}(-1)^{k+1}}_{(-1)}\sum_{i=k+1}^{n}\frac{\partial^2 a}{\partial x_i^2} dx_1\wedge\cdots\wedge dx_k$$

$$+ (-1)^{nk}\sum_{j=1}^{k}\sum_{i=k+1}^{n}(-1)^i\frac{\partial^2 a}{\partial x_j\partial x_i} dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_k\wedge dx_i$$

$$* \operatorname{sgn}(j,k+1,\dots,\hat{i},\dots n\ 1,\dots\hat{j}\dots k\ i)$$

$$(-1)^{j-i+k(n-k)} = (-1)^{n-i+k-1+n-k+j-1+k(n-k)}$$

$$\Rightarrow \mathcal{D}\, a(x)\, dx_1\wedge\cdots\wedge dx_k = -\Delta a\, dx_1\wedge\cdots\wedge dx_k.$$

f为M上Morse函数. $d_f := e^{-f/\hbar} d\, e^{f/\hbar}$. 对 d 形变为 d_f, $d_f^* = e^{f/\hbar} d^* e^{-f/\hbar}$

$$\omega\in\ker d_f \Rightarrow d(e^{f/\hbar}\omega) = 0$$

$$\Rightarrow \ker d_f = e^{-f/\hbar}\cdot\ker d$$ 理解为 $\forall\ \omega\in\ker d_f$, $\exists\ \alpha\in\ker d$ s.t. $\omega = e^{-f/\hbar}\alpha$

$$\operatorname{Im} d_f:\ \omega = d_f\alpha = e^{-f/\hbar} d(e^{f/\hbar}\alpha)$$

$$\Rightarrow \operatorname{Im} d_f = e^{-f/\hbar} \operatorname{Im} d$$

$$\Rightarrow \ker d_f / \operatorname{Im} d_f = \ker d / \operatorname{Im} d \cong H^k(M)$$

这种形变不改变上同调！

Witten算子 $\hat{H} := \frac{\hbar^2}{2}(d_f d_f^* + d_f^* d_f)$

① $(\hat{H}\alpha, \beta) = (\alpha, \hat{H}\beta)$

② $(\hat{H}\alpha, \alpha) \geq 0$

③ $\hat{H}\alpha = 0 \iff \begin{cases} d_f \alpha = 0 \\ d_f^* \alpha = 0 \end{cases}$

④ $\operatorname{Ker} \hat{H} \cong H^k(M)$

完全平行于 $\mathcal{D}$ 的证明. 下面重点来看如何把其与振子近似只联系，先改写 $\hat{H}$:

$$\hat{H} = \frac{\hbar^2}{2}\mathcal{D} + \frac{1}{2}\underbrace{\langle df, df \rangle_p}_{\text{光滑函数}} + \hbar \underbrace{R}_{\text{不带微分}}$$

R 是一个张量场

如: $R(f dx_i \wedge dx_j) = f \underbrace{R^{kl}_{ij}}_{\text{光滑依赖于x之矩阵}} dx_k \wedge dx_l$

proof: $\forall \alpha \in \Omega^k$

$$d_f \alpha = e^{-f/\hbar} d(e^{f/\hbar}\alpha)$$

$$= d\alpha + \frac{1}{\hbar} df \wedge \alpha, \quad \underbrace{\kappa_f}_{\text{对f没微分与R一样的代数算子}} \alpha := df \wedge \alpha$$

$$\Rightarrow d_f = \frac{1}{\hbar}\kappa_f + d$$

同理 $d_f^* = \frac{1}{\hbar} k_f^* + d^*$ 后面再说定义

$$\Rightarrow \hat{H} = \frac{\hbar^2}{2}\Big[\big(d + \tfrac{1}{\hbar}k_f\big)\big(d^* + \tfrac{1}{\hbar}k_f^*\big) + \big(d^* + \tfrac{1}{\hbar}k_f^*\big)\big(d + \tfrac{1}{\hbar}k_f\big)\Big]$$

$$= \frac{\hbar^2}{2}\Delta + \frac{1}{2}\left(k_f k_f^* + k_f^* k_f\right) + \hbar\cdot\frac{1}{2}\left(d k_f^* + k_f d^* + d^* k_f + k_f^* d\right)$$

$k_i \alpha := dx_i \wedge \alpha \Rightarrow k_f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} k_i$

$$k_f^* = \sum_{i=1}^n k_i^* \frac{\partial f}{\partial x_i}$$

$$\Rightarrow \frac{1}{2}\left(k_f k_f^* + k_f^* k_f\right)$$

$$= \frac{1}{2}\sum_{i,j=1}^n \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\left(k_i k_j^* + k_j^* k_i\right)$$

$k_j^* = ?$ $\quad k_j^* \alpha = \sum_{s=1}^k (-1)^{s-1} g^{j i_s} dx_{i_1}\wedge\cdots\wedge\widehat{dx_{i_s}}\cdots\wedge dx_{i_k}$

反称张量 $\alpha \equiv dx_{i_1}\wedge\cdots\wedge dx_{i_k}$

验证这一点: $(k_j^*\alpha, \beta) = (\alpha, k_j\beta)$

取 $\alpha = dx_{i_1}\wedge\cdots\wedge dx_{i_k}$, $\beta = dx_{m_1}\wedge\cdots\wedge dx_{m_{k-1}}$

$$LHS = \sum_{s=1}^k (-1)^{s-1} g^{j i_s}\left(dx_{i_1}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_{i_k}, dx_{m_1}\wedge\cdots\wedge dx_{m_{k-1}}\right)$$

$$RHS = \left(dx_{i_1}\wedge\cdots\wedge dx_{i_k}, dx_j\wedge dx_{m_1}\wedge\cdots\wedge dx_{m_{k-1}}\right)$$

$\langle dx_i, dx_j\rangle_1 = g^{ij} \Rightarrow RHS = \int \det g^{\{i\},\{j\}\cup\{m\}}\ \Omega$

$$
g = \begin{array}{c|c}
 & j \quad m_{j_1} \cdots\cdots m_k \\
\hline
\begin{matrix} i_1 \\ \vdots \\ i_k \end{matrix} & \begin{bmatrix} \times & & & \\ \times & & & \\ \times & & & \\ \times & & & \end{bmatrix}
\end{array}
\qquad s=1 \qquad (-1)^0 g^{j i_1} \langle d\hat{x}_{i_1} \wedge \cdots \wedge dx_{i_k}, \; dx_{m_1} \wedge \cdots \wedge dx_{m_{k-1}} \rangle
$$

所以左边其实是g的j列行列式展开，故 LHS=RHS

而 $(k_i k_j^* + k_j^* k_i)\, dx_{i_1} \wedge \cdots \wedge dx_k$

$$
= \sum_{s=1}^{k} (-1)^{s-1} g^{j i_s}\, dx_i \wedge dx_{i_1} \wedge \cdots \widehat{dx_{i_s}} \wedge \cdots \wedge dx_{i_k}
$$

$$
+ g^{ji}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}
$$

$$
+ \sum_{s=1}^{n} (-1)^s g^{j i_s}\, dx_i \wedge dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_s}} \wedge \cdots \wedge dx_{i_k}
$$

$$
= g^{ij}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad (\text{实线性空间 } g^{ij} = g^{ji})
$$

$$
\Rightarrow \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} (k_i k_j^* + k_j^* k_i)
$$

$$
= \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = \sum_{i,j=1}^{n} \langle dx_i, dx_j \rangle_1 \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}
$$

$$
= \sum_{i,j=1}^{n} \left\langle \frac{\partial f}{\partial x_i} dx_i, \frac{\partial f}{\partial x_j} dx_j \right\rangle_1 = \sum_{i,j=1}^{n} \langle df, df \rangle_1
$$

下面验证

$$R = \frac{1}{2}\left(k_f d^* + k_f^* d + d^* k_f + d k_f^*\right) \text{ 无微分.}$$

注意到 $d = \sum_{j=1}^{n} k_j \partial_j \qquad d^* = \sum_{j=1}^{n} \partial_j^* k_j^*$

$$R = \frac{1}{2}\sum_{i,j=1}^{n}\Big(\frac{\partial f}{\partial x_i} k_i \partial_j^* k_j^* + \frac{\partial f}{\partial x_i} k_i^* k_j \partial_j + \partial_j^* k_j^* \frac{\partial f}{\partial x_i} k_i + k_j \partial_j \frac{\partial f}{\partial x_i} k_i^*\Big)$$

∂^* 是什么? $(\partial_j \alpha, \beta) = (\alpha, \partial_j^* \beta)$

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle_p \Omega = \int_M a(x)\, b(x) \underbrace{\Omega \det g^{\{i\},\{j\}}}_{F(x)dx \in C^\infty dx}$$

$$\Rightarrow (\partial_j \alpha, \beta) = \int_M b(x) \frac{\partial a}{\partial x_j} F(x)\, dx$$

$$= -\int_M \frac{\partial b}{\partial x_j} a F(x)\, dx - \int_M b(x) a(x) \frac{\partial F}{\partial x_j} dx$$

$$= -(\alpha, \partial_j \beta) - (\alpha, A\beta)$$

无微分

$\Rightarrow \partial_j^* = -\partial_j +$ 无微分算符

于是在差一个零阶微分算符意义下 ∂^* 可替换为 ∂.

k_i 常系数 故 $[\partial^*, k_i] = 0$. 但 k_i^* 不是还涉及 g^{ij} 但交换子

$[\partial^*, k_i^*] \sim (\partial g^{ij}) \sim$ 无微分算符 (不作用于 f 上.只 g^{ij} 上)

$$\Rightarrow R \sim \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}\left(-k_i k_j^* + k_i^* k_j - k_j^* k_i + k_j k_i^*\right)\partial_j$$

$$\sim \frac{1}{2}\sum_{i,j=1} \frac{\partial f}{\partial x_i}\left(-g^{ij} + g^{ji}\right) \sim 0$$

取局部平坦坐标 $g_{ij} = \delta_{ij}$ at $(x_1, \dots, x_n)$

这时 $\partial_j^* = -\partial_j$ 而不仅 $\partial_j^* \sim -\partial_j$

这时 ∂, k, k^* 均为常系数算子（在这个正交坐标邻域）

故它们交换子精确为 0.（而不只像前面用 $\sim$）但 $[\partial_i, \frac{\partial f}{\partial x_j}]$ 非零，前面用 $\sim$，省略掉的是这些交换子，补上则有

$$R^E = \frac{1}{2}\sum_{i,j=1}^{n}\left[\frac{\partial^2 f}{\partial x_i \partial x_j} k_j^* k_i + \frac{\partial^2 f}{\partial x_i \partial x_j} k_j k_i^*\right]$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}\left(k_j k_i^* - k_i^* k_j\right)$$

现在证 Morse 定理.

首先由 Morse 引理 $\exists$ 坐标 $x_1, \dots, x_n$ 使 f 在临界点处

$$f = f(p) + \sum_{i=1}^{n} \varepsilon_i x_i^2, \quad \mathrm{Index}(p) = \#\, \varepsilon_i < 0$$

在每个点上引入局部坐标 U，且取 g_{ij} s.t. $g_{ij}|_U = \delta_{ij}$

且这个 g_{ij} 基底可延拓到整个 M 上定义，这可以用单位分解完成，则在每个临界点附近有

$$\hat{H} = -\frac{\hbar^2}{2}\Delta + \frac{1}{2}\underbrace{\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^i}}_{4|x|^2} + \frac{1}{2}\sum_{i,j=1}^{n}\underbrace{\frac{\partial^2 f}{\partial x^i \partial x^j}}_{2\varepsilon_j\delta_{ij}}(k_j k_i^* - k_i^* k_j)$$

H_0 R

$$= \underbrace{-\frac{\hbar^2}{2}\Delta + 2|x|^2}_{H_0} + \underbrace{\hbar\sum_{j=1}^{n}\varepsilon_j(k_j k_j^* - k_j^* k_j)}_{R}$$

$x=0$ 即临界点（极小值点）处，这时 $\frac{1}{2}(df, df)$ 也因 $df=0$ 为 0

考虑 H_0 项，这就是个谐振子，设略好说的

$$\psi_m = \hbar^{-n/4} f_m\left(\frac{x}{\sqrt{\hbar}}\right), \quad E_m = \sum_{j=1}^{m}\hbar\omega_j\left(m_j + \frac{1}{2}\right)$$

$$m = (m_1, m_2, \cdots, m_n), \quad f_m \xrightarrow{|y|\to\infty} e^{-|y|}$$

考虑 R 项 设 α 是 R 在 p 处本征矢（对于外微分形式）

$\Rightarrow$ $K(p)\alpha = \mu\alpha$ α 定义在 M 局部的 f 临界点附近.

取 $\omega = \hbar^{-\frac{n}{4}} f_m\left(\frac{x}{\sqrt{\hbar}}\right) e(x)\,\alpha$, $\quad e(x) := \begin{cases} 1 & |x| \le \delta_1 \\ 0 & |x| \ge \delta_2 \end{cases}$

这把 α 拓延为了 M 上定义的微分形式 $\omega \in \Omega^k(M)$

下面证局部振子近似过程：

$$\hat{H}\omega = (E_m + \hbar\mu)\omega + O(\hbar)$$

由于 $f_m \sim e^{-\frac{x}{\hbar}}$ 指数 $\ll O(\hbar)$ 只用在 x 内极小，$e(x)=1$ 内考虑问题即可，其余部分均可吸收入 $O(\hbar)$ 中

$$\hat{H}\omega = \left(-\frac{\hbar^2}{2}\Delta + 2|x|^2 + \hbar R\right)\hbar^{-\frac{n}{4}} f_m\left(\frac{x}{\sqrt{\hbar}}\right)\alpha$$

$$= E_m \hbar^{-n/2} f_m \frac{x}{\sqrt{\hbar}}\alpha + \hbar\left(R(0)+R(x)-R(0)\right)\omega$$

$$= \left(E_m + \hbar\mu\right)\omega + \hbar\left(R(x)-R(0)\right)\hbar^{-n/4} f_m\left(\frac{x}{\sqrt{\hbar}}\right)\alpha$$

用了 $R(x)$ 作用纤维方向可把 f_m 提到外面

利用谐振子近似定理的一个引理。即若 $g(x)$ 有 s 阶 $x=0$ 零点

则 $g(x)\cdot \hbar^{-n/4} f_m\left(\frac{x}{\sqrt{\hbar}}\right) = O\left(\hbar^{s/2}\right)$ 这里 $s=1$ 故

$$\hat{H}(\omega) = \left(E_m + \hbar\mu\right)\omega + O\left(\hbar^{3/2}\right)$$

多体谐振子近似定理

考虑在每个临界点处写下 Witten 算子，谐振子部分是

$$E_m^{(s)} = \sum_{j=1}^{n} \hbar\omega_j^{(s)}\left(m_j^{(s)} + \frac{1}{2}\right) + \mu^{(s)}$$

其实是2

$R(0)$ 部分

(s) 标记临界点. $\{\varepsilon^{(j)}\}$ 是 $E_m^{(s)}$ 记重数升序排列

设 λ_j 是 $\hat{H}$ 在整个流形 M 上的本征值，则 $\forall M \in \mathbb{N}$

$$\lambda_j = \varepsilon^{(j)} + O(\hbar) : j = 1, \dots, M.$$

$\forall M$ 是说对任意取定的 M, $\exists\, \hbar_0 = \hbar(M)$, s.t. 上式成立.

这两个定理和前面对 C^∞ 上 $H = -\frac{\hbar^2}{2}\Delta + V(x)$ 的两个近似定理是类似的，甚至比这个更简单

下面要利用前面说过的 Witten 复形重要性质:

$\mathrm{Ker}\,\hat{H} \cong H^k(M) \Rightarrow$ 零特征值个数 $=$ k-betti 数

又因为 $\hat{H}$ 非负定, 故这其实就是要对这个体系基态进行计数

物理上这一点是由 Witten index $W = \mathrm{Tr}(-1)^F$ 完成的. 见 Tong 讲义. 这里讲数学严格做法. 由谐振子近似理论已知。

$$\lambda_j = \varepsilon^{(j)} + O(\hbar) \quad , \quad j = 1, \cdots, M$$

$$\varepsilon^{(j)} = \sum_{k=1}^{n} \hbar(1 + 2m_k^{(j)}) + \mu^{(j)}\hbar$$

现在我们来关注 λ_j 的近似值, $\varepsilon^{(j)}$ 的零点. 注意 $\mu^{(j)}$ 是 $R|_p$ 的本征值. p 是 f 临界点.

$$R = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(k_i k_j^* - k_j^* k_i)$$

$$\underset{\text{Morse坐标}}{\overset{p点}{\sim}} \quad \sum_{j=1}^{n} \varepsilon_j (k_j k_j^* - k_j^* k_j)$$

取 $\varepsilon_j = \begin{cases} -1, & j = 1, \cdots, m \\ 1, & j = m+1, m+2, \cdots, n \end{cases}$ 其中 m 是 f 在 p 处指标

$\Rightarrow$ Morse 坐标下 $g^{ij} \sim \delta^{ij}$ 故 $R|_p \alpha$, $\alpha = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ 很容易计算.

$$k_j^* \alpha = \begin{cases} 0 & , j \notin I := (i_1 \cdots i_k) \\ (-1)^{s-1} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_s}} \wedge \cdots \wedge dx_{i_k} & , j = i_s \in I \end{cases}$$

$$k_j \alpha = dx_j \wedge \alpha$$

$\Rightarrow (k_j k_j^* - k_j^* k_j)\alpha$

① $j \notin I(\alpha)$ $= -k_j^*(dx_j \wedge \alpha) = -\alpha$

② $j \in I(\alpha)$, $j = i_s$

$k_j \alpha = 0$ 因为 $dx_{i_s} \wedge dx_{i_s} = 0$

$$k_j k_j^* \alpha = dx_{i_s} \wedge (dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_s}} \wedge \cdots \wedge dx_{i_k})(-1)^{s-1}$$

$= \alpha$

$\Rightarrow R|_p \alpha = \mu \alpha$, $\mu = -\#I\cap J - \#\bar{I}\cap\bar{J} + \#I\cap\bar{J} + \#\bar{I}\cap J$

这里 $J := \{1, 2, \cdots, m\}$, $I = \{i_1, \cdots, i_s\}$

$\Sigma^{(j)} = \sum_{j=1}^{n} \hbar(1+2m_j) + \hbar\mu_I^{(j)}$ 谐振子的能级数 m_j

↑ 这里的 m_j 是谐振子的能级数，别和 Morse 理论的 m_k 混淆

和 α 的指标集 I 互相独立取值. 故 $\Sigma^{(j)}$ 最小的是

$m_j = 0$, $I = J$ 的情况 这时若

① $m \neq k$. 此种情况不会发生 $I = J$ (由②可知这意味着对 $\Sigma^{(j)} = 0$ 无基态)

② $m = k$. 有且仅有一种可能的 α 且 $\mu = -n$

$\Rightarrow \Sigma^{(j)} = 0$ 对应 j 标记临界点. 故每个 $m=k$ 的临界点贡献一个 $\Sigma^{(j)} = 0$

大于0的指

$$\Rightarrow \Sigma^{(j)} = \underbrace{0, 0, 0 \cdots 0}_{m=k\text{ 的临界点个数，即 } m_k} \cdots\cdots$$

而 $\varepsilon^{(j)}$ 是"精确"为 $O(\hbar)$ 的本征值 λ_j 精确为 0 的本征值值

$\dim\ker H = \dim H^k(M;\mathbb{R}) = \underline{b_k \leq m_k}$

↑ 弱 Morse 不等式得证.

$$T := \frac{\hbar}{\sqrt{2}}(d_f + d_f^*) \Rightarrow \hat{H} = T^2 \Rightarrow [T,\hat{H}] = 0$$

$$\Omega = \Omega^+ \oplus \Omega^-, \quad \Omega^+ := \bigoplus_{k\in \text{even}} \Omega^k, \quad \Omega^- := \bigoplus_{k\in \text{odd}} \Omega^k$$

$$T : \Omega^+ \rightleftharpoons \Omega^-$$

前面已说明 $\hat{H}$ 有 b_k 个基态. 有 m_k 个 $O(\hbar)$ 基态. 则有 $m_k - b_k$ 个 $O(\hbar)$ 激发态, 这 $m_k - b_k$ 个态张成一个子空间 M_k. s.t. $\hat{H}$ 在其上无核.(只在 $O(\hbar)$ 精度上有, 因只全部)

由于 $[T,\hat{H}]=0$ 则 $\forall\ |a\rangle \in M_k$, $\hat{H}T|a\rangle = T\hat{H}|a\rangle \sim O(\hbar)T|a\rangle$

而 $T|a\rangle \in \Omega_{k-1}\oplus\Omega_{k+1}$ 故 $T|a\rangle \in M_{k-1}\oplus M_{k+1}$

即 $T: M_k \to M_{k-1}\oplus M_{k+1}$ 且 $\ker T = 0$. 否则. $\exists\, |t\rangle \in \ker T$

$T^2|t\rangle = H|t\rangle = 0 \neq O(\hbar)$

$$M^+ := \bigoplus_{k\in\text{even}} M^k, \quad M^- := \bigoplus_{k\in\text{odd}} M^k.$$

则 $T: M^+ \leftrightarrow M^-$, 类似. $T: \Omega^+ \leftrightarrow \Omega^-$

T 在 M^+, M^- 上作用时均无核. 故 T 是同构 则.

$$\dim M^+ = \dim M^-.$$

$\dim M^+ = m_0 - b_0 + m_2 - b_2 + \cdots$

$\dim M^- = m_1 - b_1 + m_3 - b_3 + \cdots$

T 是超对称性算符

Fermion $\leftrightarrow$ Boson.

这立刻得到 Morse 指标定理

$$\sum_{k=0}^{n} (-1)^k m_k = \sum_{k=0}^{n} (-1)^k b_k$$

$$M_k^+ := M_0 \oplus M_2 \oplus \cdots \oplus M_k$$

$$M_k^- := M_1 \oplus M_3 \oplus \cdots \oplus M_{k+1}$$

$k \in$ even.

$T: M_k^+ \to M_k^-$. $\ker T|_{M_k^+} = 0 \Rightarrow \dim M_k^+ \leq \dim M_k^-$

而 $\dim M_k^+ = m_0 - b_0 + m_2 - b_1 + \cdots + m_k - b_k$

$\dim M_k^- = m_1 - b_1 + m_3 - b_3 + \cdots + m_{k+1} - b_{k+1}$

这立刻得到强Morse不等式:

$$\sum_{j=0}^{k} (-1)^{k+1-j} m_j \geq \sum_{j=0}^{k} (-1)^{k+1-j} b_j \quad k \in \text{even}$$

同理取 $k \in$ odd 可同样得到 $k \in$ odd 的强不等式.

至此, 我们完全证明了 Morse 定理

Remark: 振子近似定理可推广到相空间上的周期轨道上的复向量丛去考虑. 其思路有助于研究复的 Morse 理论即 Picard-Lefschetz 理论, 目前还有待研究.