An Introduction to Boundary Conformal Field Theory

with an application to the Ising–TIM interfaces

Bufan Zheng

Department of Physics, The University of Tokyo

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Outline

1. Brief Review of BCFT
2. Interface Between Ising and TIM
3. Conclusion

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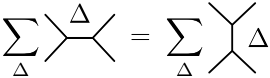
2. Interface Between Ising and TIM
3. Conclusion

Motivation

- BPZ approach [DMS97]
    - Φ_i, Δ_i, s_i, C_{ijk}, $\mathcal{W}_c$
    - non-linear constraints



    - Computationally convenient on S^2
- Segal's approach [Seg04]
    - TQFT-like construction: $\Sigma \mapsto \mathcal{H}_\Sigma$
    - Geometrical sewing conditions
    - Computationally inconvenient, but it makes the world-sheet formulation manifest: CFT can be defined on arbitrary (punctured) Riemann surfaces, including those with boundaries.
- Boundaries arise naturally in many physical contexts
    - String theory: D-branes, O-planes, $\cdots$
    - Quantum gravity: FZZT/ZZ branes [FZZ00; ZZ01], AdS/BCFT [Tak11]
    - CMT: Kondo effect [AL91; GLW21], Fermi-edge singularity [AL94]

Half-plane approach: gluing condition

Conformal transformations should preserve the boundary: $f(x \in \mathbb{R}) \in \mathbb{R}$

- Global conformal symmetry: $SL(2,\mathbb{C}) \to SL(2,\mathbb{R})$
- No energy escapes through the boundary:

$$T_{xy}(x,0) = 0 \iff T(x) = \bar{T}(\bar{x})\big|_{x\in\mathbb{R}}$$

For a general chiral algebra $\mathcal{W}$ ($\overline{\mathcal{W}} \cong \mathcal{W}$), the following gluing condition holds:

$$\boxed{W(z) = \Omega \bar{W}(\bar{z}), \quad \forall z = \bar{z}} \tag{1}$$

$\Omega : \mathcal{W} \to \mathcal{W}$ is an automorphism; for the Vir algebra, $\Omega = \mathrm{id}$. This automorphism defines a new action of $\mathcal{W}$ on $\mathcal{H}_i$:

$$\pi_i^{\Omega}(W)\,|h\rangle := \pi_i(\Omega W)\,|h\rangle$$

As a $\mathcal{W}$-module, $\pi_i^{\Omega} \curvearrowright \mathcal{H}_i \cong \pi_i \curvearrowright \mathcal{H}_{\omega(i)}$.

Doubling trick for currents

By the Schwarz reflection principle, we can analytically continue $W(z)$ from the UHP to $\mathbb{C}$:

$$W_{\text{doubled}}(z) := \begin{cases} W(z), & \operatorname{Im} z > 0 \\ \Omega \bar{W}(\bar{z}), & \operatorname{Im} z < 0 \end{cases}$$

The semicircle mode integrals can be extended to full-circle integrals:

$$\begin{aligned} W_n^{(H)} :&= \frac{1}{2\pi i} \int_{\frown} dz\, z^{n+h_w-1} W(z) - \frac{1}{2\pi i} \int_{\frown} d\bar{z}\, \bar{z}^{n+h_w-1} \Omega \bar{W}(\bar{z}) \\ &= \frac{1}{2\pi i} \int_{\circlearrowleft} dz\, z^{n+h_w-1} W_{\text{doubled}}(z) \end{aligned}$$

Thus, only half of the symmetry is preserved:

$$\mathcal{W}[W_n] \times \overline{\mathcal{W}}[\bar{W}_n] \to \mathcal{W}^{\text{diag}}[W_n^{(H)}]$$

Doubling trick for correlators I

Correlators of bulk fields $\Phi\left(|i,\bar{\imath}\rangle\,;z,\bar{z}\right)$ satisfy the following Ward identity:

$$W_{\text{doubled}}(w)\Phi\left(|i,\bar{\imath}\rangle\,;z,\bar{z}\right)\sim$$
$$\sum_{n>-h_w}\left(\frac{1}{(w-z)^{n+h_w}}\Phi\left(W_n\,|i\rangle\otimes|\bar{\imath}\rangle\,;z,\bar{z}\right)+\frac{1}{(w-\bar{z})^{n+h_w}}\Phi\left(|i\rangle\otimes\Omega\bar{W}_n\,|\bar{\imath}\rangle\,;z,\bar{z}\right)\right)$$

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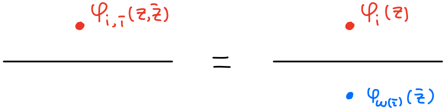
Compare this with the chiral Ward identity in $\mathbb{C}$:

$$W^{(P)}(w)\Phi\left(|i,\bar{\imath}\rangle\,;z,\bar{z}\right)\sim\sum_{n>-h_w}\frac{1}{(w-z)^{n+h_w}}\Phi\left(W_n\,|i\rangle\otimes|\bar{\imath}\rangle\,;z,\bar{z}\right)$$

We can treat $\bar{z}$ as independent of z, then $\varphi_{i,\bar{\imath}}$ splits into $\varphi_i(z)$ and $\varphi_{\omega(\bar{\imath})}(\bar{z})$.



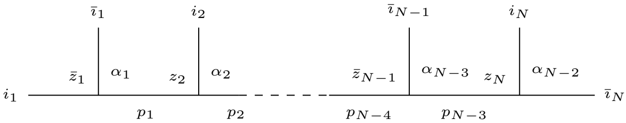
$$W_{\text{double}}(w)\varphi_{i,\bar{\imath}}(z,\bar{z})\sim$$

$$W^{(P)}(w)\left(\varphi_i(z)\varphi_{\omega(\bar{\imath})}(\tilde{z})\right)\Big|_{\tilde{z}=\bar{z}}$$

Doubling trick for correlators II

Thus, $\langle \Phi_{i_1,\bar{\imath}_1}(z_1,\bar{z}_1)\cdots\Phi_{i_n,\bar{\imath}_n}(z_n,\bar{z}_n)\rangle_{\mathsf{UHP}}$ can be expanded in chiral conformal blocks of the bulk CFT (e.g., in the half-ladder channel):

$$\mathcal{F}_{\mathbf{i}}^{\mathbf{p},\boldsymbol{\alpha}}(z) =$$



For example, a 2-point function in the UHP is like a chiral 4-point function on the plane:

$$\langle \Phi_{i_1,\bar{\imath}_1}(z_1,\bar{z}_1)\Phi_{i_2,\bar{\imath}_2}(z_2,\bar{z}_2)\rangle_{\mathsf{UHP},\alpha} = \sum_p \mathsf{B}_p^\alpha \mathcal{F}_p(z_1,\bar{z}_1,z_2,\bar{z}_2)$$

Note that

$$\langle \Phi_{i_1,\bar{\imath}_1}(z_1,\bar{z}_1)\Phi_{i_2,\bar{\imath}_2}(z_2,\bar{z}_2)\Phi_{i_3,\bar{\imath}_3}(z_3,\bar{z}_3)\Phi_{i_4,\bar{\imath}_4}(z_4,\bar{z}_4)\rangle_{\mathbb{C}} = \sum_{p,\bar{p}} \mathsf{C}_{p\bar{p}}\mathcal{F}_p(z_1,z_2,z_3,z_4)\bar{\mathcal{F}}_{\bar{p}}(\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4)$$

Doubling trick for correlators II

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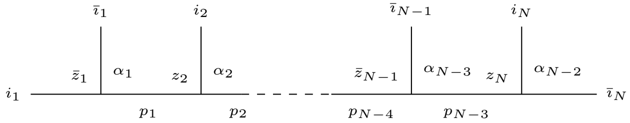

For example, a 2-point function in the UHP is like a chiral 4-point function on the plane:

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Also,

$$\langle \Phi_{i,\bar{\imath}}(z,\bar{z})\rangle_{\mathsf{UHP},\alpha} \sim \langle \phi_i(z)\phi_i(\bar{z})\rangle_{\mathbb{C}} = \frac{A_{i\bar{\imath}}^{\alpha}}{|z-\bar{z}|^{h_i+h_{\bar{\imath}}}}\delta_{\bar{\imath},\omega^{-1}(i^+)} \neq 0$$

Boundary fields: BCPOs

From the bulk one-point function, we find a singularity as a bulk field approaches the boundary, indicating additional boundary-localized states. This is encoded in the bulk-boundary OPE:

$$\varphi_{i,\bar{\imath}}(z,\bar{z}) = \sum_k C^{\alpha}_{(i\bar{\imath})k} |z-\bar{z}|^{h_k - h_i - h_{\bar{\imath}}} \psi_k^{\alpha\alpha}(\mathrm{Re}\, z).$$

- $\psi_k^{\alpha\alpha}(x)$ corresponds to a state on a semicircle, rather than a circle in the bulk.
- $\psi_k^{\alpha\alpha}(x)$ is localized on the boundary and cannot be moved into the bulk.
- BCPOs furnish the open-channel Hilbert space, which transforms under one copy of $\mathcal{W}$; the bulk Hilbert space transforms under $\mathcal{W} \times \overline{\mathcal{W}}$:

$$\mathcal{H}^{(H)}_{\alpha\alpha} = \bigoplus_i \mathcal{H}_i^{\oplus n^i_\alpha}, \quad \mathcal{H}^P = \bigoplus_{i,\bar{\jmath}} \left(\mathcal{H}_i \otimes \bar{\mathcal{H}}_{\bar{\jmath}}\right)^{\oplus M_{i\bar{\jmath}}}$$

Boundary fields: BCPOs

Boundary fields themselves have an OPE:

$$\psi_k^{\alpha\alpha}(x_1)\psi_l^{\alpha\alpha}(x_2) = \sum_m C_{klm}^{\alpha}(x_1 - x_2)^{h_m - h_k - h_l}\psi_m^{\alpha\alpha}(x_2), \qquad x_1 > x_2.$$

- A pair is mutually local when the analytic continuation is unique:

$$\psi_2(x_2)\psi_1(x_1)\big|_{x_2>x_1} \xrightarrow{\text{unique a.c.}} \psi_1(x_1)\psi_2(x_2)\big|_{x_1>x_2}.$$

- Unlike bulk fields, generic boundary fields need not be mutually local: exchanging their order may produce non-trivial monodromy.
- Ordering matters! For an open-string disk amplitude, the moduli space integral decomposes into a sum over chambers corresponding to inequivalent cyclic orderings.
- The currents $W(x)$ are local with respect to all boundary (and bulk) fields and regular on the boundary: $W(z) = W(x) + O(z - \bar{z})$.

Boundary fields: BCCOs

Different boundary conditions may be imposed on different parts of the boundary. A jump $\beta \to \alpha$ produces a singularity, which is represented by a boundary-condition-changing operator $\psi_k^{\alpha\beta}(x)$.

- $\psi_k^{\alpha\beta}(x)$ corresponds to an open-string state stretched between the branes α and β:

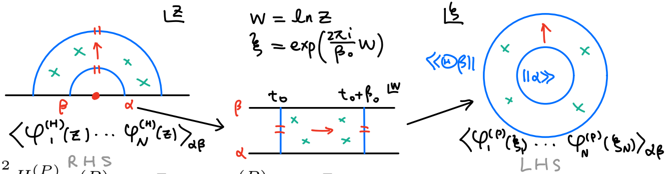
$$\mathcal{H}_{\alpha\beta} = \bigoplus_i n_{\alpha\beta}^i \mathcal{H}_i.$$

- For $\alpha \neq \beta$, there is no $SL(2,\mathbb{R})$-invariant vacuum: $n_{\alpha\beta}^0 = 0$.
- BCCOs also have an OPE:

$$\psi_k^{\alpha\beta}(x_1)\psi_l^{\gamma\delta}(x_2) = \delta_{\beta\gamma}\sum_m C_{klm}^{\alpha\beta\delta}(x_1 - x_2)^{h_m - h_k - h_l}\psi_m^{\alpha\delta}(x_2), \quad x_1 > x_2$$

- In contrast, the BCPO sector $\mathcal{H}_{\alpha\alpha}$ may contain several copies of the vacuum $\mathbf{1}$. In this talk, we restrict to $n_{\alpha\alpha}^0 = 1$, i.e., to a fundamental D-brane. [GRW01]

Boundary state formalism



$$\langle\langle\Theta\alpha\| e^{-\frac{2\pi^2}{\beta_0}H^{(P)}}\varphi_1^{(P)}(\xi_1,\bar{\xi}_1)\cdots\varphi_N^{(P)}(\xi_N,\bar{\xi}_N)\|\alpha\rangle\rangle := \mathcal{J}(\xi,z)\ \mathrm{Tr}_{\mathcal{H}_\alpha^{(H)}}\left[e^{-\beta_0 H^{(H)}}\varphi_1^{(H)}(z_1,\bar{z}_1)\cdots\varphi_N^{(H)}(z_N,\bar{z}_N)\right].$$

Here $\langle\langle\Theta\beta| := (\Theta|\beta\rangle\rangle)^\dagger$, where Θ is the anti-unitary CPT operator arising from orientation reversal. $\langle\langle\Theta\cdot\|$ is the BPZ conjugation. (See Appendix) Some authors [SZ24] instead define boundary states on the disk (zero temperature), rather than via the annulus (finite temperature):

$$\langle 0|\varphi_1^{(P)}(\zeta_1,\bar{\zeta}_1)\cdots\varphi_N^{(P)}(\zeta_N,\bar{\zeta}_N)\|\alpha\rangle\rangle_\Omega := \mathcal{J}(\zeta,z)\ \left\langle\varphi_1^{(H)}(z_1,\bar{z}_1)\cdots\varphi_N^{(H)}(z_N,\bar{z}_N)\right\rangle_{\mathrm{UHP},\alpha}$$

Here $\zeta = \frac{1-iz}{1+iz}$. They differ by an overall factor.

Ishibashi states

From the boundary-state definition, choose $\varphi_N^{(H)}(z,\bar{z}) = W^{(H)}(z) - \Omega\bar{W}^{(H)}(\bar{z})$. Since it vanishes at $z=\bar{z}$, the boundary state obeys the gluing condition:

$$\boxed{\left(W_n^{(P)} - (-1)^{h_W}\,\Omega\bar{W}_{-n}^{(P)}\right)\|\alpha\rangle\!\rangle_\Omega = 0}$$

(see Appendix).

Untwisted: $\Omega = \mathrm{id}$

For each irreducible $\mathcal{W}$-module $\mathcal{H}_i$, the gluing condition is solved by the Ishibashi state

$$\boxed{\|i\rangle\!\rangle = \sum_{N=0}^{\infty} |i,N\rangle \otimes U|i,N\rangle \in \widehat{\mathcal{H}_i \otimes \overline{\mathcal{H}}_{i+}}}$$

Here U is the anti-unitary operator satisfying

$$UW_n = (-1)^{h_W} W_n U, \qquad U|i,N\rangle \in \bar{\mathcal{H}}_{i+}.$$

Twisted: general Ω

The gluing automorphism is implemented by a unitary map

$$V_\Omega : \mathcal{H}_{\omega(i)} \longrightarrow \mathcal{H}_i,$$
$$\pi_i^\Omega(A) = V_\Omega \pi_{\omega(i)}(A) V_\Omega^{-1}.$$

The corresponding twisted Ishibashi state is

$$\boxed{\|i\rangle\!\rangle_\Omega = (\mathrm{id}\otimes V_\Omega)\|i\rangle\!\rangle \in \widehat{\mathcal{H}_i \otimes \overline{\mathcal{H}}_{\omega^{-1}(i+)}}}$$

Non-linear constraints: Cardy condition

From the finite-temperature definition of a boundary state:

$$Z^{\mathrm{cl}}_{\alpha\beta}(\tilde q) := \langle\!\langle \Theta\beta \| \tilde q^{\frac{1}{2}\left(L_0^{(P)}+\bar L_0^{(P)}-\frac{c}{12}\right)} \|\alpha\rangle\!\rangle = \mathrm{Tr}_{\mathcal{H}^{(H)}_{\alpha\beta}}\, q^{\,L_0^{(H)}-\frac{c}{24}} =: Z^{\mathrm{op}}_{\alpha\beta}(q)$$

with $q := e^{-2\pi i\tau}$, $\tilde q := e^{-2\pi i/\tau}$. By $\Theta\|\alpha\rangle\!\rangle = \sum_i \overline{B^{\,i}_{\alpha}}\,\Theta\|i\rangle\!\rangle := \sum_i \overline{B^{\,i}_{\alpha}}\,\|i^+\rangle\!\rangle$, and the Ishibashi-state overlap $\langle\!\langle j\|q^{L_0-\frac{c}{24}}\|i\rangle\!\rangle = \chi_i^{\mathcal{W}}\delta_{ij}$, we obtain

$$Z^{\mathrm{cl}}_{\alpha\beta}(\tilde q) = \sum_i B^{\,i^+}_{\beta} B^{\,i}_{\alpha}\chi_i(\tilde q), \qquad Z^{\mathrm{op}}_{\alpha\beta}(q) = \sum_i n^{\,i}_{\alpha\beta}\chi_i^{\mathcal{W}}(q)$$

Since $\chi_j^{\mathcal{W}}(\tilde q) = \sum_i S_{ji}\chi_i^{\mathcal{W}}(q)$, open–closed duality gives

$$\boxed{\sum_{i,j} B^{\,j^+}_{\beta} B^{\,j}_{\alpha} S_{ji}\chi_i^{\mathcal{W}}(q) = \sum_i n^{\,i}_{\alpha\beta}\chi_i^{\mathcal{W}}(q), \qquad n^{\,i}_{\alpha\beta} \in \mathbb{Z}_{\geq 0}}$$

Non-linear constraints: Cardy condition

$$\boxed{\sum_{i,j} B_{\beta}^{j^+} B_{\alpha}^{j} S_{ji} \chi_i^{\mathcal{W}}(q) = \sum_i n_{\alpha\beta}^{i} \chi_i^{\mathcal{W}}(q) \qquad n_{\alpha\beta}^{i} \in \mathbb{Z}_{\geq 0}}$$

- Verlinder formulae implies a simple solution: Cardy state
$$\|a\rangle\rangle = \sum_i \frac{S_{ai}}{\sqrt{S_{0i}}} \|i\rangle\rangle$$
- We keep the χ_i on both sides, since they are not linearly independent in general; e.g., $\chi_{i^+} = \chi_i$.
- Due to Θ, the LHS has the structure SBB^+ rather than $SB\bar{B}$. Without Θ, one would obtain
$$\tilde{Z}^{\mathrm{cl}}_{\alpha\beta}(t) = \langle\langle \beta \| e^{-\pi t H_{\mathrm{cl}}} \| \alpha \rangle\rangle = \sum_i \bar{B}_{\beta}^{i} B_{\alpha}^{i} \chi_i(q).$$
The two coincide only if one additionally assumes $B^{i^+} = \bar{B}^i$ [Beh+00; SS03].
- In general, this subtlety cannot be ignored: relaxing this assumption can yield additional consistent boundary states [GRW02], but it has been overlooked in parts of the literature [CMW23].

Non-linear constraints: genus-zero sewing

- (Boundary)^4: $\left\langle \psi_i^{\alpha\beta}(x_1)\psi_j^{\beta\gamma}(x_2)\psi_k^{\gamma\delta}(x_3)\psi_\ell^{\delta\alpha}(x_4)\right\rangle$, $x_1 > x_2 > x_3 > x_4$

$$\sum_p C^{\alpha\beta\gamma}_{ijp}\, C^{\gamma\delta\alpha}_{klp^+}\, C^{\alpha\gamma\alpha}_{p^+p0}\, F_{pq}\begin{bmatrix} j & k \\ i & l^+ \end{bmatrix} = C^{\beta\gamma\delta}_{jkq}\, C^{\alpha\beta\delta}_{iql^+}\, C^{\alpha\delta\alpha}_{l^+l0}$$

- Bulk–(Boundary)^2: $\left\langle \phi_{\mathrm{i}}(z,\bar{z})\psi_k^{\alpha\beta}(x_1)\psi_\ell^{\beta\alpha}(x_2)\right\rangle$, $x_1 > x_2$

$$C^{\alpha}_{\mathrm{i},q}\, C^{\alpha\beta\beta}_{kql^+}\, C^{\alpha\beta\alpha}_{l^+l0} = \sum_{p,m} e^{i\pi\Delta_{p,m}}\, F_{p^+m}\begin{bmatrix} \bar{\imath} & l \\ i^+ & k \end{bmatrix} F_{mq}\begin{bmatrix} i & \bar{\imath} \\ k^+ & l \end{bmatrix} C^{\beta}_{\mathrm{i},p}\, C^{\alpha\beta\alpha}_{klp^+}\, C^{\alpha\alpha\alpha}_{pp^+0}$$

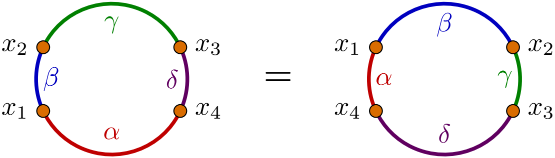
$$\Delta_{p,m} := 2h_m - 2h_i - h_k - h_l + \frac{1}{2}(h_p + h_q).$$

- (Bulk)^2–Boundary: $\langle \phi_{\mathrm{i}}(z,\bar{z})\phi_{\mathrm{j}}(w,\bar{w})\psi_k^{\alpha\alpha}(x)\rangle$

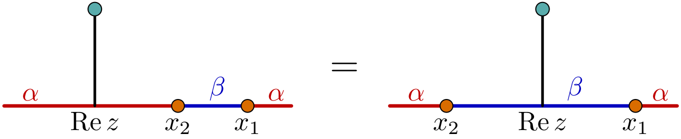
$$C_{\mathrm{ijm}}\, C^{\alpha}_{\mathrm{m},k} = \sum_{p,q,r} e^{i\frac{\pi}{2}\Delta_{p,q,r}}\, F_{qr}\begin{bmatrix} k & \bar{\jmath} \\ p^+ & j \end{bmatrix} F_{p^+m}\begin{bmatrix} \bar{\imath} & r \\ i & j \end{bmatrix} F_{r\bar{m}}\begin{bmatrix} \bar{\imath} & \bar{\jmath} \\ m & k \end{bmatrix} C^{\alpha}_{\mathrm{i},p}\, C^{\alpha}_{\mathrm{j},q}\, C^{\alpha\alpha\alpha}_{pqk^+}$$

$$\Delta_{p,q,r} := h_k + h_p - h_q - 2h_r + h_m - h_{\bar{m}} - h_i + h_{\bar{\imath}} + h_j + h_{\bar{\jmath}}.$$

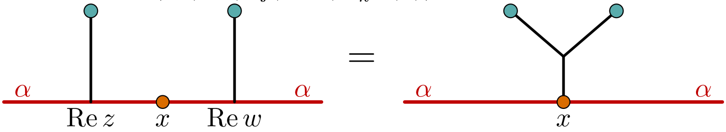
- (Boundary)^4: $\left\langle \psi_i^{\alpha\beta}(x_1)\psi_j^{\beta\gamma}(x_2)\psi_k^{\gamma\delta}(x_3)\psi_\ell^{\delta\alpha}(x_4)\right\rangle$, $x_1 > x_2 > x_3 > x_4$



- Bulk–(Boundary)^2: $\left\langle \phi_{\mathrm{i}}(z,\bar{z})\psi_k^{\alpha\beta}(x_1)\psi_\ell^{\beta\alpha}(x_2)\right\rangle$, $x_1 > x_2$



- (Bulk)^2–Boundary: $\langle \phi_{\mathrm{i}}(z,\bar{z})\phi_{\mathrm{j}}(w,\bar{w})\psi_k^{\alpha\alpha}(x)\rangle$



Non-linear constraints: cluster condition

A useful weaker consequence of the $(\text{Bulk})^2$–Boundary sewing relation is the cluster condition. For a fundamental D-brane, set $\psi_0^{\alpha\alpha} = \mathbf{1}$ and $\bar{\imath} = \omega^{-1}(i^+)$, $\bar{\jmath} = \omega^{-1}(j^+)$. Projecting onto the vacuum channel gives

$$\boxed{C^\alpha_{\mathrm{i},0} C^\alpha_{\mathrm{j},0} = \sum_{\mathrm{k}} F_{k0} \begin{bmatrix} j & \bar{\jmath} \\ i & \bar{\imath} \end{bmatrix} C_{\mathrm{ijk}}\, C^\alpha_{\mathrm{k},0} := \sum_{\mathrm{k}} \Xi_{\mathrm{ijk}}\, C^\alpha_{\mathrm{k},0}}$$

Using $C^\alpha_{\mathrm{i},0} = A^\alpha_{\mathrm{i}} / A^\alpha_0$, this becomes

$$A^\alpha_{\mathrm{i}} A^\alpha_{\mathrm{j}} = \sum_{\mathrm{k}} \Xi_{\mathrm{ijk}}\, A^\alpha_0 A^\alpha_{\mathrm{k}}$$

In practice, this condition is much simpler than the full genus-zero sewing relations and is often imposed on its own. For Cardy boundary states,

$$\Xi_{\mathrm{ijk}} = \left(\frac{S_{00} S_{0k}}{S_{0i} S_{0j}} \right)^{\frac{1}{2}} N_{ij}^k$$

More fun with orientifolds

- A crosscap state $\|C\rangle\!\rangle$ corresponds to an O-plane
- Gluing condition $\left(W_n^i - (-1)^n(-1)^{h_i}\,\overline{W}{}^i_{-n}\right)\|C\rangle\!\rangle = 0$
- The Ishibashi state $\|\mathcal{C}_i\rangle\!\rangle = e^{\pi i(L_0 - h(\phi_i))}\|\mathcal{B}_i\rangle\!\rangle$
- Overlap amplitudes

$$\mathcal{Z}^{\mathcal{K}(O,O)}_{\text{loop}}(\tau,\bar\tau) = \mathrm{Tr}_{\mathcal{H}\otimes\bar{\mathcal{H}}}\left(\Omega_O\, q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0-\frac{\bar c}{24}}\right) \quad \xleftrightarrow[\tau=4it]{l=\frac{1}{4t}} \quad \mathcal{Z}^{\mathcal{K}(O,O)}_{\text{tree}}(l) = \langle\!\langle \Theta C\| e^{-2\pi l\left(L_0+\bar L_0-\frac{c+\bar c}{24}\right)}\|C\rangle\!\rangle$$

$$\mathcal{Z}^{\mathcal{M}(O,D)}_{\text{loop}}(t) = \mathrm{Tr}_{\mathcal{H}_B}\left(\Omega_O\, e^{-2\pi t\left(L_0-\frac{c}{24}\right)}\right) \quad \xleftrightarrow{l=\frac{1}{8t}} \quad \mathcal{Z}^{\mathcal{M}(O,D)}_{\text{tree}}(l) = \langle\!\langle \Theta C\| e^{-2\pi l\left(L_0+\bar L_0-\frac{c+\bar c}{24}\right)}\|B\rangle\!\rangle.$$

The $\mathcal{Z}^{\mathcal{M}}$ amplitudes are expanded in hatted characters:

$$\widehat{\chi}(\tau) = e^{-\pi i\left(h-\frac{c}{24}\right)}\chi\left(\tau+\frac{1}{2}\right)$$

Coefficient of each character need not be non-negative because of Ω_O.

- As a consistent string background, the tadpole should be cancelled.
- Further reading: [BP09; RS13]

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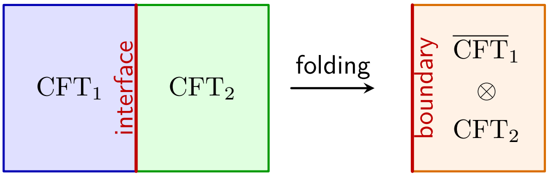
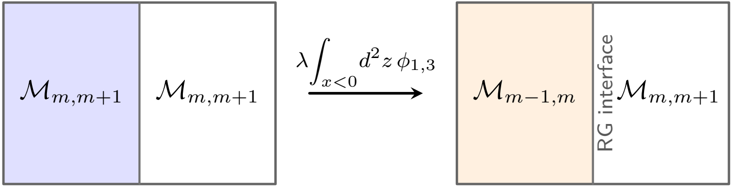
Folding trick



- If both theories are Virasoro RCFTs, the folded theory remains rational with respect to $\mathrm{Vir}_1 \times \mathrm{Vir}_2$. Hence, boundaries preserving $T_1 = \bar{T}_1, T_2 = \bar{T}_2$ are the well-understood *rational branes*.
- For interfaces, however, we are more interested in boundaries preserving only the diagonal Virasoro algebra, $T_1 + T_2 = \bar{T}_1 + \bar{T}_2$. Since c_{tot} can exceed 1, this generally becomes an irrational BCFT classification problem and is much more difficult.

Gaiotto's RG domain wall: setup

Consider the RG flow between two adjacent minimal models:



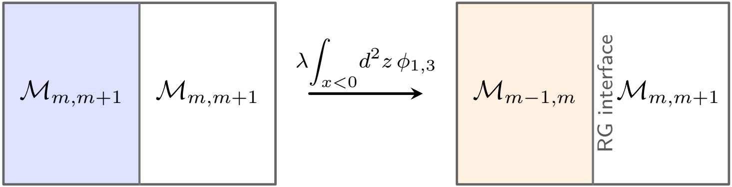
Gaiotto suggests the following state: [Gai12]

$$\| RG \rangle\!\rangle = \sum_{t,s,r}^{s-t \in 2\mathbb{Z}} \frac{\sqrt{S^{(k-1)}_{1,t} \, S^{(k+1)}_{1,s}}}{S^{(k)}_{1,r}} \, \| A, t, r, s \| t, s; \widetilde{B}, Z_2 \rangle\!\rangle$$

Some aspects have been checked numerically in [CFG24].

Gaiotto's RG domain wall: setup

Consider the RG flow between two adjacent minimal models:



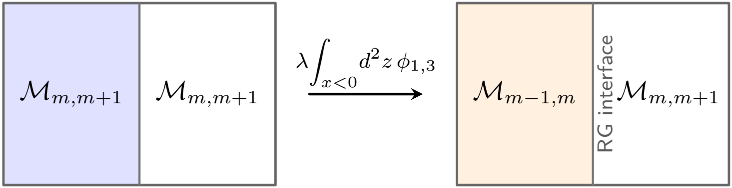
Gaiotto suggests the following state: [Gai12]

$$\|RG\rangle\rangle = \sum_{t,s,r}^{s-t\in 2\mathbb{Z}} \frac{\sqrt{S^{(k-1)}_{1,t}\;S^{(k+1)}_{1,s}}}{S^{(k)}_{1,r}} \|A,t,r,s\|t,s;\widetilde{B},Z_2\rangle\rangle$$

A boundary $\|\widetilde{\mathcal{B}}\rangle\rangle := \sum_{t,s}^{s-t\in 2\mathbb{Z}} \sqrt{S^{(k-1)}_{1,t}S^{(k+1)}_{1,s}}\,\|t,s;\widetilde{B},Z_2\rangle\rangle$

Gaiotto's RG domain wall: setup

Consider the RG flow between two adjacent minimal models:



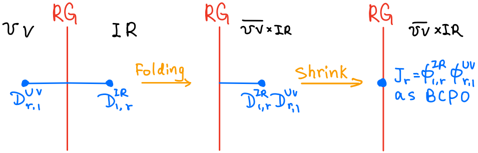
Gaiotto suggests the following state: [Gai12]

$$\|RG\rangle\rangle = \sum_{t,s,r}^{s-t\in 2\mathbb{Z}} \frac{\sqrt{S^{(k-1)}_{1,t}\ S^{(k+1)}_{1,s}}}{S^{(k)}_{1,r}} \|A,t,r,s\| t,s;\widetilde{B},Z_2\rangle\rangle$$

An interface $\mathcal{I}_1 := \sum_{t,r,s} \|A,t,r,s\|/S^{(k)}_{1,r}$.

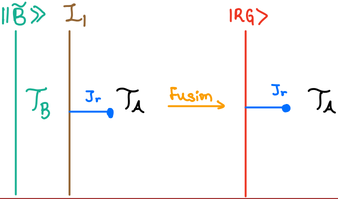
Gaiotto's RG domain wall: construction

- TDLs preserved by the RG flow enhance the boundary symmetry:



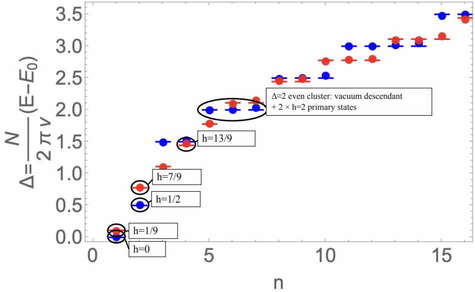
$$\mathcal{A} \xrightarrow{\text{bosonic ext.}} \mathcal{B} \xrightarrow{\text{fermionic } \mathbb{Z}_2 \text{ ext.}} \widetilde{\mathcal{B}}$$

- Choose a rational brane $\|\widetilde{B}\rangle\rangle \in \mathcal{T}_{\mathcal{B}}$ preserving the enhanced symmetry. An interface $\mathcal{I}_1$ projects it back to the original theory and is chosen so that the relevant TDLs can end on it, producing $\|\mathrm{RG}\rangle\rangle$.



New interface between Ising and TIM

Antunes & Rong found the following lattice spectrum [AR26]:



$$
\begin{aligned}
Z^{\text{op}}_{\text{AR}}(\tilde q) =&\chi^{W_3}_{h=0}(\tilde q) + \chi^{W_3}_{h=1/2}(\tilde q)\\
&+ 2\chi^{W_3}_{h=2}(\tilde q) + \chi^{W_3}_{h=1/9}(\tilde q)\\
&+ \chi^{W_3}_{h=7/9}(\tilde q) + \chi^{W_3}_{h=13/9}(\tilde q)
\end{aligned}
$$

This suggests that the boundary symmetry is enhanced to $\mathcal{W}_3$. Indeed,

$$
c_{\mathcal{W}_3(5,6)} = c_{\text{Ising}} + c_{\text{TIM}} = 1.2 > 1
$$

This differs from Gaiotto's state:

$$
\begin{aligned}
Z^{\text{op}}_{\text{RG}}(\tilde q) =&3\chi_1^{\text{Ising}}(\tilde q)\chi_1^{\text{TIM}}(\tilde q) + 4\chi_\sigma^{\text{Ising}}(\tilde q)\chi_{\sigma'}^{\text{TIM}}(\tilde q) + \chi_\epsilon^{\text{Ising}}(\tilde q)\chi_1^{\text{TIM}}(\tilde q)\\
&+ \chi_1^{\text{Ising}}(\tilde q)\chi_{\epsilon''}^{\text{TIM}}(\tilde q) + 3\chi_\epsilon^{\text{Ising}}(\tilde q)\chi_{\epsilon''}^{\text{TIM}}(\tilde q)
\end{aligned}
$$

Are there any other models?

An RCFT may become irrational with respect to a smaller algebra:

- $SU(2)_1$ WZW: $\widehat{\mathfrak{su}(2)}_1 \to \mathsf{Vir}_{c=1}$ [GRW02]
- $SU(2)_k$ WZW: $\widehat{\mathfrak{su}(2)}_k \to \mathsf{PF}_k \times \mathsf{Vir}$ [BR09]
- Compact boson at an (ir)rational radius: $\widehat{\mathfrak{u}(1)} \to \mathsf{Vir}_{c=1}$ [GR01; Jan01]
- Ising $\times$ Ising: $\mathsf{Vir}_{c=\frac{1}{2}} \times \mathsf{Vir}_{c=\frac{1}{2}} \to \mathsf{Vir}_{c=1}$ [OA97]
- $\mathsf{TIM}_f \times \mathsf{TIM}_f$: $\mathsf{SVir}^{\mathcal{N}=1}_{c=\frac{7}{10}} \times \mathsf{SVir}^{\mathcal{N}=1}_{c=\frac{7}{10}} \to \mathsf{SVir}^{\mathcal{N}=1}_{c=\frac{7}{5}}$ [GY08]
- Some general arguments on $\mathcal{M} \times \mathcal{M}$: $\mathsf{Vir}_c \times \mathsf{Vir}_c \to \mathsf{Vir}_{2c}$ [KRW09; BRR21]

  $\ldots$

A generic boundary of Ising $\times$ TIM preserves the diagonal $\mathsf{Vir}_{c=1.2}$. We want to find more boundaries whose symmetry is enhanced to $\mathcal{W}_3$.

Are there any more solutions?

$$Z^{\text{op}} = n_1\chi_0^{\mathcal{W}_3} + n_2\chi_{\frac{1}{9}}^{\mathcal{W}_3} + n_3\chi_{\frac{7}{9}}^{\mathcal{W}_3} + n_4\chi_2^{\mathcal{W}_3} + n_5\chi_{\frac{1}{2}}^{\mathcal{W}_3} + n_6\chi_{\frac{13}{9}}^{\mathcal{W}_3} + n_7\chi_{\frac{3}{5}}^{\mathcal{W}_3} + n_8\chi_{\frac{2}{45}}^{\mathcal{W}_3} + n_9\chi_{\frac{17}{45}}^{\mathcal{W}_3} + n_{10}\chi_{\frac{1}{10}}^{\mathcal{W}_3} + n_{11}\chi_{\frac{32}{45}}^{\mathcal{W}_3} + n_{12}\chi_{\frac{8}{5}}^{\mathcal{W}_3}$$

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.

Are there any more solutions?

$$Z^{\mathrm{cl}} = \tilde{n}_1\chi_0^{\mathcal{W}_3} + \tilde{n}_2\chi_{\frac{1}{9}}^{\mathcal{W}_3} + \tilde{n}_3\chi_{\frac{7}{9}}^{\mathcal{W}_3} + \tilde{n}_4\chi_2^{\mathcal{W}_3} + \tilde{n}_5\chi_{\frac{1}{2}}^{\mathcal{W}_3} + \tilde{n}_6\chi_{\frac{13}{9}}^{\mathcal{W}_3} + \tilde{n}_7\chi_{\frac{3}{5}}^{\mathcal{W}_3}$$
$$+ \tilde{n}_8\chi_{\frac{2}{45}}^{\mathcal{W}_3} + \tilde{n}_9\chi_{\frac{17}{45}}^{\mathcal{W}_3} + \tilde{n}_{10}\chi_{\frac{1}{10}}^{\mathcal{W}_3} + \tilde{n}_{11}\chi_{\frac{32}{45}}^{\mathcal{W}_3} + \tilde{n}_{12}\chi_{\frac{8}{5}}^{\mathcal{W}_3}$$

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.
- S-transform to the closed channel:

Are there any more solutions?

$$Z^{\mathrm{cl}} = \tilde{n}_1 \chi_0^{\mathcal{W}_3} + \cancel{\tilde{n}_2 \chi_{\frac{1}{9}}^{\mathcal{W}_3}} + \cancel{\tilde{n}_3 \chi_{\frac{7}{9}}^{\mathcal{W}_3}} + \tilde{n}_4 \chi_2^{\mathcal{W}_3} + \tilde{n}_5 \chi_{\frac{1}{2}}^{\mathcal{W}_3} + \cancel{\tilde{n}_6 \chi_{\frac{13}{9}}^{\mathcal{W}_3}} + \tilde{n}_7 \chi_{\frac{3}{5}}^{\mathcal{W}_3}$$
$$+ \cancel{\tilde{n}_8 \chi_{\frac{2}{45}}^{\mathcal{W}_3}} + \cancel{\tilde{n}_9 \chi_{\frac{17}{45}}^{\mathcal{W}_3}} + \tilde{n}_{10} \chi_{\frac{1}{10}}^{\mathcal{W}_3} + \cancel{\tilde{n}_{11} \chi_{\frac{32}{45}}^{\mathcal{W}_3}} + \cancel{\tilde{n}_{12} \chi_{\frac{8}{5}}^{\mathcal{W}_3}}$$

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.
- S-transform to the closed channel:
  - $\|B\rangle\rangle \in \widehat{\mathcal{H}}_{\mathrm{cl}}$: only bulk Ising $\times$ TIM states may contribute to Z^{cl}.

Are there any more solutions?

$$Z^{\mathrm{op}} = \chi_0 + n_2\chi_{\frac{1}{9}} + n_2\chi_{\frac{7}{9}} + 2\chi_2 + n_5\chi_{\frac{1}{2}} + n_2\chi_{\frac{13}{9}} + 2n_{12}\chi_{\frac{3}{5}} + n_{11}\chi_{\frac{2}{45}} + n_{11}\chi_{\frac{17}{45}} + n_{10}\chi_{\frac{1}{10}} + n_{11}\chi_{\frac{32}{45}} + n_{12}\chi_{\frac{8}{5}}, \qquad n_1 = 1$$

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.
- S-transform to the closed channel:
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Are there any more solutions?

$$Z^{\text{cl}} = \sum_i \tilde{n}_i \chi_i^{\mathcal{W}_3} = \sum_i b_i \chi_i^{\mathsf{Vir}}, \qquad b_i \geq 0 \quad \Longrightarrow \quad 65 \text{ candidate sets } \{n_i\} \text{ remain.}$$

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.
- S-transform to the closed channel:
  - $\|B\rangle\rangle \in \widehat{\mathcal{H}}_{\text{cl}}$: only bulk Ising $\times$ TIM states may contribute to Z^{cl}.
  - Vir preserved $\Rightarrow$ positivity in the χ^{Vir} expansion.

Are there any more solutions?

$$Z^{\mathrm{op}}_{\alpha\beta} = \mathbf{M} \cdot \boldsymbol{\chi}^{\mathcal{W}_3} \xrightarrow{S} \mathcal{S} \cdot \widetilde{\mathbf{M}} \cdot \boldsymbol{\chi}^{\mathsf{Vir}} = \sum_h \overline{B^h_\alpha} B^h_\beta \chi^{\mathsf{Vir}}_h = Z^{\mathrm{cl}}_{\alpha\beta},$$

$G^h := \left(\overline{B^h_\alpha} B^h_\beta\right) \succeq 0$ and higher-rank analogues $\Longrightarrow$ 8 compatible Z^{op} with $Z^{\mathrm{op}}_{\mathrm{AR}}$.

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.
- S-transform to the closed channel:
  - $\|B\rangle\rangle \in \widehat{\mathcal{H}}_{\mathrm{cl}}$: only bulk Ising $\times$ TIM states may contribute to Z^{cl}.
  - Vir preserved $\Rightarrow$ positivity in the χ^{Vir} expansion.
- Overlap amplitudes:
  - Assuming $B^{i^+} = \overline{B^i}$ and that overlap amplitudes admit a $\chi^{\mathcal{W}_3}$ expansion, positivity of the Gram matrix can help.
  - If we admit a nontrivial Ω, 8 more compatible Z^{op} arise.

Are there any more solutions?

$$\|B\rangle\rangle = \sum_{i=0}^{N} B_i \, \|i; \mathcal{W}_3\rangle\rangle \text{ ✗}, \qquad \|B\rangle\rangle = \sum_{i=0}^{\infty} B_i \, \|i; \mathsf{Vir}\rangle\rangle \text{ ✓}$$

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.
- S-transform to the closed channel:
  - $\|B\rangle\rangle \in \widehat{\mathcal{H}}_{\mathrm{cl}}$: only bulk Ising $\times$ TIM states may contribute to Z^{cl}.
  - Vir preserved $\Rightarrow$ positivity in the χ^{Vir} expansion.
- Overlap amplitudes:
  - Assuming $B^{i^+} = \overline{B^i}$ and that overlap amplitudes admit a $\chi^{\mathcal{W}_3}$ expansion, positivity of the Gram matrix can help.
  - If we admit a nontrivial Ω, 8 more compatible Z^{op} arise.
- From Z^{op} to $\|B\rangle\rangle$:
  - $\mathcal{W}_3$ is an enhanced symmetry, so the most general ansatz is a Virasoro Ishibashi expansion. However, the expansion is infinite.

Are there any more solutions?

$$A_i^\alpha A_j^\alpha = \sum_k \Xi_{ijk} \, A_0^\alpha A_k^\alpha$$

- The open-string sector is a Vir-module with accidental $\mathcal{W}_3$ enhancement, giving the general ansatz.
- S-transform to the closed channel:
  - $\|B\rangle\rangle \in \widehat{\mathcal{H}}_{\mathrm{cl}}$: only bulk Ising $\times$ TIM states may contribute to Z^{cl}.
  - Vir preserved $\Rightarrow$ positivity in the χ^{Vir} expansion.
- Overlap amplitudes:
  - Assuming $B^{i^+} = \overline{B^i}$ and that overlap amplitudes admit a $\chi^{\mathcal{W}_3}$ expansion, positivity of the Gram matrix can help.
  - If we admit a nontrivial Ω, 8 more compatible Z^{op} arise.
- From Z^{op} to $\|B\rangle\rangle$:
  - $\mathcal{W}_3$ is an enhanced symmetry, so the most general ansatz is a Virasoro Ishibashi expansion. However, the expansion is infinite.
  - The cluster condition may help us.

1. Brief Review of BCFT

2. Interface Between Ising and TIM

3. Conclusion

Conclusion

- The classification of boundaries in RCFTs is largely under control.
- For defects/interfaces, the folding trick doubles the central charge and can turn the boundary problem irrational, so a general classification remains challenging.
- RG interfaces between adjacent minimal models admit a heuristic construction.
- Many more general interfaces, including symmetry-enhanced ones, remain to be discovered and classified.

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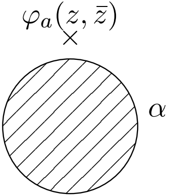
4 Appendix

Why Θ? I

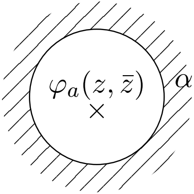
We use the conformal transformation $z \mapsto -\frac{1}{z}$ to define the BPZ conjugate [Pol07; 伊藤 11] of $|\text{in}\rangle$ and construct the corresponding $\langle\text{out}|$:

$$|a\rangle = \lim_{z\to 0} \varphi_a(z,\bar{z})\,|0\rangle \to {}_{\mathsf{bpz}}\langle a| = \sum_b \lim_{z\to\infty} z^{2h_a}\,\bar{z}^{2\bar{h}_a}\,\langle 0|\,g^{ab}\varphi_a(z,\bar{z})$$

Here $g_{ab} := \lim_{z\to 0} z^{2h_a}\,\bar{z}^{2\bar{h}_a}\,\langle\varphi_a(z,\bar{z})\varphi_b(0,0)\rangle \propto \delta_{ab^+}$. This is different from Hermitian conjugation. ${}_{\mathsf{bpz}}\langle\cdot|\cdot\rangle$ is bilinear, unlike the Hermitian inner product $\langle\cdot|\cdot\rangle$. Similarly, $z \mapsto -\frac{1}{z}$ relates *in* and *out* boundary states:



$= \langle 0|\varphi_a(z,\bar{z})\|\alpha\rangle\!\rangle$



$= {}_{\mathsf{bpz}}\langle\!\langle\alpha\|\varphi_a(z,\bar{z})|0\rangle$

Why Θ? II

$$\text{LHS} = \lim_{z=\infty} z^{2h_a} \bar{z}^{2\bar{h}_a} \langle 0| \varphi_a(z,\bar{z}) \|\alpha\rangle\!\rangle = \sum_{b,i} B^i_\alpha g_{ab\ \mathsf{bpz}}\langle b\|i\rangle\!\rangle$$

$$\text{RHS} = \lim_{z\to 0} {}_{\mathsf{bpz}}\langle\!\langle \alpha\|\varphi_a(z,\bar{z})\,|0\rangle = {}_{\mathsf{bpz}}\langle\!\langle \alpha\|a\rangle$$

Comparing the two expressions and using ${}_{\mathsf{bpz}}\langle i|j\rangle = (-1)^F{}_{\mathsf{bpz}}\langle j|i\rangle$, we obtain

$$_{\mathsf{bpz}}\langle\!\langle \alpha\| = \sum_{i,j} B^i_\alpha g_{ij} \left(\|j\rangle\!\rangle\right)^\dagger \sim \left(\Theta\|\alpha\rangle\!\rangle\right)^\dagger$$

Here $\sim$ denotes equality up to a possible sign. If we normalize $\Theta\,|i\rangle = |i^+\rangle$ as we do in this talk, there is no ambiguity.

Another advantage of the zero-temperature definition of boundary states is that boundary coefficients are directly related to one-point functions:

$$B^{i^+}_\alpha = A^\alpha_{i\bar{i}}, \quad \left(\Theta\,|i\rangle = |i^+\rangle\right)$$

Gluing condition in boundary state formalism

On the UHP, conformal invariance requires $T^{(H)}(z) = \bar{T}^{(H)}(\bar{z})\big|_{z=\bar{z}}$. In the boundary state definition, choose the insertion $T^{(H)}(z) - \bar{T}^{(H)}(\bar{z})$, which therefore vanishes on the boundary inside arbitrary correlators. Under the map to the ξ-plane, $\xi = e^{\frac{2\pi i}{\beta_0}\ln z}, \bar{\xi} = e^{-\frac{2\pi i}{\beta_0}\ln \bar{z}}$, positive real $z = \bar{z}$ is mapped to $|\xi| = 1$, where

$$\bar{\xi} = \xi^{-1}, \qquad \frac{d\bar{\xi}/d\bar{z}}{d\xi/dz} = -\bar{\xi}^2, \qquad \{\xi, z\} = \{\bar{\xi}, \bar{z}\}.$$

The Schwarzian terms therefore cancel, giving

$$\left[T^{(P)}(\xi) - \xi^{-4}\bar{T}^{(P)}(\xi^{-1})\right] \|\alpha\rangle\!\rangle = 0.$$

Using

$$T^{(P)}(\xi) = \sum_n L_n^{(P)} \xi^{-n-2}, \qquad \bar{T}^{(P)}(\bar{\xi}) = \sum_n \bar{L}_n^{(P)} \bar{\xi}^{-n-2},$$

and matching powers of ξ, we obtain the Virasoro Ishibashi condition

$$\left(L_n^{(P)} - \bar{L}_{-n}^{(P)}\right) \|\alpha\rangle\!\rangle = 0, \qquad n \in \mathbb{Z}.$$